MONOCHROMATIC SUBGRAPHS IN RANDOMLY COLORED DENSE MULTIPLEX NETWORKS

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Abstract. Given a sequence of graphs $\{G_n\}_{n\geq 1}$ and fixed graph H, denote by $T(H, G_n)$ the number of monochromatic copies of the graph H in G_n in a uniformly random c_n-coloring of the vertices of G_n. In this paper we study the joint distribution of monochromatic subgraphs for dense multiplex networks, that is, networks with multiple layers. Specifically, we consider the joint distribution of $\boldsymbol{T}_n := (T(H, G_n), T(H', G'_n))$, for two sequences of dense graphs $\{G_n\}_{n\geq 1}$ and $\{G'_n\}_{n\geq 1}$ on the same set of vertices and two fixed graphs H and H'.

Under a new notion of joint convergence of the graphs G_n and G'_n in the cut metric, we show that when the number of $c_n = c$ is fixed, the limiting distribution of $\boldsymbol{T}_n$ is the sum of two independent components, one of which is a bivariate Gaussian and the other is a sum of bivariate stochastic integrals. This generalizes the classical birthday problem, which involves understanding the asymptotics of $T(K_s, K_n)$, the number of monochromatic s-cliques in a complete graph K_n (s-matching birthdays among a group of n friends), to general monochromatic subgraphs in multiplex networks. This also extends previous results on the marginal convergence of $T(H, G_n)$ and is useful in establishing the joint convergence of various subgraph counting statistics that arise from random vertex coloring of graphs. Several examples are discussed and an alternate proof using random matrix theory is also presented for the case of monochromatic edges.

1. Introduction

Let $G_n = (V(G_n), E(G_n))$ be a sequence of graphs with vertex set $V(G_n) = [n] := \{1, 2, \cdots, n\}$ and edge $E(G_n)$. Suppose the vertices of G_n are colored uniformly at random with $c \geq 2$ colors, that is,

$$\mathbb{P}(v \in [n] \text{ is assigned with color } a \in \{1, 2, \ldots, c\}) = \frac{1}{c},$$

independently from the other vertices. Given such a coloring an edge $(u, v) \in E(G_n)$ is said to be monochromatic if both u and v are assigned the same color. Formally, if X_v denotes the color of the vertex $v \in [n]$, then $(u, v) \in E(G_n)$ is monochromatic if $X_u = X_v$.

$$T(K_2, G_n) = \frac{1}{2} \sum_{1\leq u\neq v\leq n} a_{uv}(G_n)\mathbf{1}\{X_u = X_v\}, \tag{1.1}$$ eq:T_K2_G

where $A(G_n) = ((a_{uv}(G_n)))_{1\leq u,v\leq n}$ is the adjacency matrix of G_n. More generally, for a fixed connected graph $H = (V(H), E(H))$ denote by $T(H, G_n)$ the number of monochromatic copies of H in G_n, that is,

$$T(H, G_n) := \frac{1}{|Aut(H)|} \sum_{\boldsymbol{s}\in[n]_{|V(H)|}} \prod_{(a,b)\in E(H)} a_{s_a s_b}(G_n)\mathbf{1}\{X_{=\boldsymbol{s}}\} \tag{1.2}$$ def:T_H_G

where:

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- $[n]_{|V(H)|}$ is the set of all $|V(H)|$-tuples $\boldsymbol{s} = (s_1, \cdots, s_{|V(H)|}) \in [n]^{|V(H)|}$ with distinct indices.[1] Thus, the cardinality of $[n]_{|V(H)|}$ is $\frac{n!}{(n-|V(H)|)!}$
- For any $\boldsymbol{s} = (s_1, \cdots, s_{|V(H)|}) \in [n]_{|V(H)|}$,
$$\mathbf{1}\{X_{=\boldsymbol{s}}\} := \mathbf{1}\{X_{s_1} = \cdots = X_{s_{|V(H)|}}\}.$$
- $Aut(H)$ is the *automorphism group* of H, that is, the number of permutations σ of the vertex set $V(H)$ such that $(x, y) \in E(H)$ if and only if $(\sigma(x), \sigma(y)) \in E(H)$.

Note that
$$\mathbb{E}T(H, G_n) = \frac{N(H, G_n)}{c^{|V(H)|-1}},$$
where $N(H, G_n)$ is the number of copies of H in G_n.

Various problems in combinatorial probability, nonparametric statistics, and theoretical computer science involve the study of (1.1) and (1.2). The following are some examples:

Example 1.1 (Birthday problem)**.** Suppose G_n is a friendship-graph (two people are connected by an edge in the graph if they are friends) which is colored uniformly randomly with $c = 365$ colors (where the colors correspond to birthdays and the birthdays are assumed to be uniformly distributed across the year). In this case a monochromatic edge in G_n corresponds to two friends with the same birthday. The celebrated birthday problem asks for the probability that there are two friends with the same birthday, that is, $\mathbb{P}(T(K_2, G_n) > 0)$. This is a celebrated problem in elementary probability (see [23, 25, 45] references therein), which appear in many different contexts, such as the study of coincidences [26], graph coloring problems [2, 10, 18], testing discrete distributions [4, 27, 54], security of hash functions [5, 60], and the discrete logarithm problem [7, 29, 34], among others. A natural generalization of the birthday problem is to consider higher-order birthday matches (often referred to as multi-collisions); that is, the number of r-tuples of friends sharing the same birthday [26, 44, 47, 61]. This corresponds to studying the asymptotics of $T(K_r, G_n)$, the number of monochromatic r-cliques K_r in the friendship graph G_n [9, 11].

Example 1.2 (Graph-based nonparametric 2-sample tests)**.** One of the fundamental problems in statistical inference is to decide whether two different data sets are generated from the same statistical model. This is the classical 2-sample problem which can be formally stated as follows: Given independent and identically distributed samples $\mathscr{X}_n = \{X_1, X_2, \ldots, X_n\}$ and $\mathscr{Y}_n = \{Y_1, Y_2, \ldots, Y_n\}$ from two multivariate distributions F and G, respectively, the two- sample problem is to test the hypothesis:
$$H_0 : F = G \quad \text{versus} \quad H_1 : F \neq G. \tag{1.3}$$ eq:FG

A theme that has emerged repeatedly in the development of nonparametric 2-sample tests is the use of random geometric graphs. This includes the celebrated Friedman-Rafsky test based on the minimum spanning free (MST) [28], tests based on nearest-neighbor (NN) graphs [30, 58], and optimal matchings [56]. Here, the idea is to construct a random geometric graph $\mathscr{G}_n$ of the pooled sample $\mathscr{X}_n \cup \mathscr{Y}_n$, and reject the null hypothesis H_0 in (1.3) if the number of edges in $\mathscr{G}$ with one end-point in sample 1 and other in sample 2 is 'small'. In other words, H_0 is rejected when the number of *non-monochromatic* edges in $\mathscr{G}$ is 'small' (where the sample labels corresponds to the colors), which is equivalent to rejecting H_0 when the number of monochromatic edges is 'large'. Hence, understanding the null distribution of such a test statistic entails studying the asymptotic properties of $T(K_2, \mathscr{G}_n)$ with $c = 2$ colors.

[1]For a set S, the set S^N denotes the N-fold cartesian product $S \times S \times \cdots \times S$.

Example 1.3 (Quadratic Rademacher chaos)**.** When $c = 2$, $T(K_2, G_n)$ centered by its mean can be expressed as a quadratic form in Rademacher variables as follows:

$$\begin{aligned} T(K_2, G_n) - \mathbb{E}T(K_2, G_n) &= \frac{1}{2} \sum_{1 \leq u \neq v \leq n} a_{uv}(G_n) \left(\mathbf{1}\{X_u = X_v\} - \frac{1}{2} \right) \\ &= \frac{1}{4} \sum_{1 \leq u \neq v \leq n} a_{uv}(G_n)(2\mathbf{1}\{X_u = 1\} - 1)(2\mathbf{1}\{X_v = 1\} - 1) \\ &= \frac{1}{4} \sum_{1 \leq u \neq v \leq n} a_{uv}(G_n) Y_u Y_v, \end{aligned} \tag{1.4}$$

eq:K2quadratic

where $Y_u = 2\mathbf{1}\{X_u = 1\} - 1$, for $1 \leq u \leq n$, are i.i.d. Rademacher random variables. Hence, when $c = 2$, $T(K_2, G_n) - \mathbb{E}T(K_2, G_n)$ is the *quadratic Rademacher chaos* [15, 52], which appears in various contexts, for example, the Hamiltonian of the Ising model [3, 8], testing independence in auto-regressive models [6], and Ramsey theory [36]. A characterization of all distributional limits of (1.4) has been obtained recently in [13].

A number of recent papers have studied the limiting distribution of $T(H, G_n)$ (appropriately centered and scaled), beginning with the work of Bhattacharya et al. [10] which considered the case $H = K_2$. Specifically, [10, Theorem 1.3] shows that $T(K_2, G_n)$ is asymptotically normal whenever the fourth moment of a suitably normalized version of $T(K_2, G_n)$ converges to 3 (the fourth moment of the standard normal distribution). This is an instance of the celebrated fourth-moment phenomenon, which was originally discovered in the seminal papers [49, 53] and has, since then, emerged as a unifying principle governing the asymptotic normality of random multilinear forms (see the book [50] for further details). Going beyond monochromatic edges, subsequently it was shown that the fourth-moment phenomenon continues to hold for the asymptotic normality of $T(K_3, G_n)$ (the number of monochromatic triangles) when $c \geq 5$ [12] and for $T(H, G_n)$, for any fixed graph H, when $c \geq 30$ [22]. For $c = 2$ a more refined result about the asymptotic normality of $T(H, G_n)$ has been obtained recently by Mani and Mikulincer [41] in terms of local influence-based conditions. While the aforementioned results provide sufficient conditions (some which are also necessary) for the asymptotic normality of $T(H, G_n)$, there are many instances where $T(H, G_n)$ has a non-Gaussian limit. This, in particular, is often the case when G_n is a sequence dense graph, that is, $|E(G_n)| = \Theta(n^2)$. In this case, to describe the limiting distribution of $T(H, G_n)$ it is convenient to adopt the framework of graph limit theory [38] and assume G_n converges (in the cut-distance) to a graphon W. Then limiting distribution of $T(H, G_n)$ (suitably standardized) has both Gaussian and non-Gaussian components, where non-Gaussian component is a (possibly) infinite weighted sum of independent centered chi-squared random variables with the weights determined by the spectral properties of a graphon derived from W (see [10, Theorem 1.4] and [9, Theorem 1.3]).

Given the above results a natural next step is to study the joint distribution of a finite collection of monochromatic subgraphs. Specifically, given a collection of $r \geq 1$ graphs $\mathcal{H} = \{H_1, H_2, \ldots, H_r\}$ what is limiting joint distribution of $(T(H_1, G_n), T(H_2, G_n), \ldots, T(H_r, G_n))$ (suitably standardized)? More generally, one can consider the joint distribution of

$$\boldsymbol{T}(\mathcal{H}, \boldsymbol{G}_n) := (T(H_1, G_n^{(1)}), T(H_2, G_n^{(2)}), \ldots, T(H_r, G_n^{(d)})),$$

where $\boldsymbol{G}_n = (G_n^{(1)}, G_n^{(2)}, \ldots, G_n^{(d)})$ is a collection of $d \geq 1$ graphs having the same vertex set $[n]$. Multiple graphs sharing a common vertex set are known as multilayer networks (or multiplexes). The ubiquitous presence of complex relational data with many interdependencies have propelled the rapidly developing literature on multiplex networks (see [1, 14, 19, 24, 33, 35, 37, 39, 40, 59] and the references therein). In this paper we derive the asymptotic distribution of $T(\mathcal{H}, \boldsymbol{G}_n)$ for a sequence of dense multiplexes $\boldsymbol{G}_n$. To describe the limiting distribution we define a natural notion of convergence of multiplexes (in the joint cut-distance) and also invoke the framework of multiple stochastic integrals (for capturing the non-Gaussian dependencies between the different marginals). Under the assumption that the multiplex $\boldsymbol{G}_n$ and a certain pairwise overlap function converges, we show that the limiting distribution of $T(\mathcal{H}, \boldsymbol{G}_n)$ has 2 independent components: one of which is a multivariate Gaussian and the other is a sum of independent bivariate stochastic integrals (see Theorem 1.1). The stochastic integrals are with respect to the same underlying Brownian motion on $[0, 1]$, which captures the dependence between the different marginals. A key ingredient of the proof is a invariance principle which allows us to replace the coloring indicators with appropriately chosen Gaussian variables (see Section 2).

1.1. **Convergence of Graphs and Multiplexes.** The theory of graph limits [16, 17, 38] has received phenomenal attention over the last few years. It builds a bridge between combinatorics and analysis, and has found applications in several disciplines including statistical physics, probability, and statistics [3, 20, 21]. For a detailed exposition of the theory of graph limits refer to Lovász [38]. Here we mention the basic definitions about the convergence of graph sequences. If F and G are two graphs, then define the homomorphism density of F into G by

$$t(F, G) := \frac{|\hom(F, G)|}{|V(G)|^{|V(F)|}},$$

where $|\hom(F, G)|$ denotes the number of homomorphisms of F into G. In fact, $t(F, G)$ is the proportion of maps $\phi : V(F) \to V(G)$ which define a graph homomorphism.

To define the continuous analogue of graphs, consider $\mathscr{W}$ to be the space of all measurable functions from $[0, 1]^2$ into $[0, 1]$ which are symmetric, that is, $W(x, y) = W(y, x)$, for all $x, y \in [0, 1]$. For a simple graph F with $V(F) = \{1, 2, \dots, |V(F)|\}$, let

$$t(F, W) = \int_{[0,1]^{|V(F)|}} \prod_{(i,j)\in E(F)} W(x_i, x_j) \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_{|V(F)|} = \mathbb{E}\left(\prod_{(i,j)\in E(F)} W(U_i, U_j) \right),$$

where $U_1, U_2, \dots, U_n$ are i.i.d. Unif$[0, 1]$. It is easy to verify that $t(F, G) = t(F, W^G)$, where W^G is the *empirical graphon* associated with the graph G which defined as:

$$W^G(x, y) = \mathbf{1}\{(\lceil |V(G)|x\rceil, \lceil |V(G)|y\rceil) \in E(G)\}. \tag{1.5}$$?eq:emp_graph?

(In other words, to obtain the empirical graphon W^G from the graph G, partition $[0, 1]^2$ into $|V(G)|^2$ squares of side length $1/|V(G)|$, and let $W^G(x, y) = 1$ in the (i, j)-th square if $(i, j) \in E(G)$, and 0 otherwise.)

The basic definition of graph limit theory is in terms of the convergence of the homomorphism densities [16, 17, 38]. Specifically, we say a sequence of graphs $\{G_n\}_{n\geq 1}$ *converges to a graphon* W if for every finite simple graph F,

$$\lim_{n\to\infty} t(F, G_n) = t(F, W). \tag{1.6}$$ eq:graph_limit

This convergence can be captured through the cut-distance which is defined as follows:

:Wconvergence⟩? **Definition 1.1.** [38] The *cut-distance* between two graphons $W_1, W_2 \in \mathcal{W}$ is defined as,

$$||W_1 - W_2||_\square := \sup_{A,B\in[0,1]} \left| \int_{A\times B} (W_1(x, y) - W_2(x, y)) \,\mathrm{d}x\mathrm{d}y \right|, \tag{1.7}$$ eq:Wconvergence

for $W_1, W_2 \in \mathscr{W}$.

Using the cut distance, one can an equivalence relation on $\mathscr{W}$ as follows: $W_1 \sim W_2$ when $\inf_\phi \|W_1 - W_2^\phi\|_\square = 0$, where the infimum taken over all measure-preserving bijections $\phi : [0,1] \to [0,1]$ and $W_1^\phi(x,y) := W_1(\phi(x), \phi(y))$, for $x, y \in [0,1]$. The orbit of $W \in \mathscr{W}$ is the set of all functions W^ϕ, as ϕ varies over all measure preserving bijections from $[0,1] \to [0,1]$. Denote by $\tilde{W}$ the closure of the orbit of W in $(\mathscr{W}, \|\cdot\|_\square)$. The quotient space $\tilde{\mathscr{W}} = \{\tilde{W} : W \in \mathscr{W}\}$ of closed equivalence classes is associated with the *cut-metric*:

$$\delta_\square(\tilde{W}_1, \tilde{W}_2) := \inf_\phi \|W_1 - W_2^\phi\|_\square.$$

The central result in graph limit theory is that a sequence of graphs G_n is converges to W in the sense of (1.6) if and only if it is a Cauchy sequence in the $\delta_\square$ metric. Moreover, a sequence of graphs $\{G_n\}_{n\geq 1}$ is to a graphon W if and only if $\delta_\square(\tilde{W}^{G_n}, \tilde{W}) \to 0$ (see [, Theorem 3.8]).

In this paper we deal with sequences of multilayer graphs (multiplexes), where one has multiple graphs on the same set vertices. To define convergence of a sequence of d-multiplexes $\{\boldsymbol{G}_n\}_{n\geq 1}$ common vertex set $V(\boldsymbol{G}_n) = [n]$, we need the notion of joint cut metric. Towards this, denote by $\mathscr{W}^d$ the d-fold Cartesian product of $\mathscr{W}$. The elements of $\mathscr{W}^d$ will be referred to as *d-graphons*. We can define an equivalence relation on $\mathscr{W}^d$ as follows: Given $\boldsymbol{W} = (W_1, W_2, \ldots, W_d) \in \mathscr{W}^d$ and $\boldsymbol{W'} = (W_1', W_2', \ldots, W_d') \in \mathscr{W}^d$, we say $\boldsymbol{W_1} \sim \boldsymbol{W_2}$, if $\inf_\phi \sum_{j=1}^d \|W_j^\phi - W_j'\|_\square = 0$, where, as before, the infimum taken over all measure-preserving bijections $\phi : [0,1] \to [0,1]$. Denote by $\tilde{\boldsymbol{W}}$ the closure of the orbit of $\boldsymbol{W}$ and the quotient space of closed equivalence classes as $\tilde{\mathscr{W}}^d = \{\tilde{\boldsymbol{W}} : W \in \mathscr{W}^d\}$. For $\tilde{\boldsymbol{W}}, \tilde{\boldsymbol{W}}' \in \tilde{\mathscr{W}}^d$, define the *joint cut-metric* as:

$$\delta_\square(\tilde{\boldsymbol{W}}, \tilde{\boldsymbol{W}}') := \inf_\phi \sum_{j=1}^d \|W_j^\phi - W_j'\|_\square.$$

We will say a sequence of d-multiplex $\{\boldsymbol{G_n}\}_{n\geq 1} = \{(G_n^{(1)}, G_n^{(2)}, \ldots, G_n^{d})\}_{n\geq 1}$ converges to d-graphon $\boldsymbol{W} = (W_1, W_2, \ldots, W_d) \in \mathscr{W}^d$, if

$$\delta_\square(\tilde{\boldsymbol{W}}^{G_n}, \tilde{\boldsymbol{W}}) \to 0, \tag{1.8}$$ eq:distanceW

where $\boldsymbol{W}^{G_n} = (W^{G_n^{(1)}}, W^{G_n^{(2)}}, \ldots, W^{G_n^{(d)}})$.

1.2. **Statement of the Results.** With the above preparations, we are now ready to state our main results. To this end, fix an integer $d \geq 1$ and a finite collection of graphs $\mathcal{H} = \{H_1, H_2, \ldots, H_d\}$. Then for a sequence of d-multiplexes $\{\boldsymbol{G_n}\}_{n\geq 1} = \{(G_n^{(1)}, G_n^{(2)}, \ldots, G_n^{d})\}_{n\geq 1}$, define the vector of standardized monochromatic copies as:

$$\boldsymbol{\Gamma}(\mathcal{H}, \boldsymbol{G}_n) = \begin{pmatrix} \Gamma(H^{(1)}, G_n^{(1)}) \\ \vdots \\ \Gamma(H^{(d)}, G_n^{(d)}) \end{pmatrix}, \tag{1.9}$$ eq:GammaHGnd

where

$$\Gamma(H, G_n) = |Aut(H)| c^{|V(H)|-\frac{3}{2}} \left\{ \frac{T(H, G_n) - \mathbb{E}(T(H, G_n))}{n^{|V(H)|-1}} \right\}, \tag{1.10}$$ eq:GammaHGn

for any graph $H = (V(H), E(H))$ and any graph sequence $\{G_n\}_{n\geq 1}$ with $G_n = (V(G_n), E(G_n))$. Our aim is to derive the limiting distribution of $\boldsymbol{\Gamma}(\mathcal{H}, \boldsymbol{G}_n)$ when $\{\boldsymbol{G_n}\}_{n\geq 1}$ converges to a d-multiplex

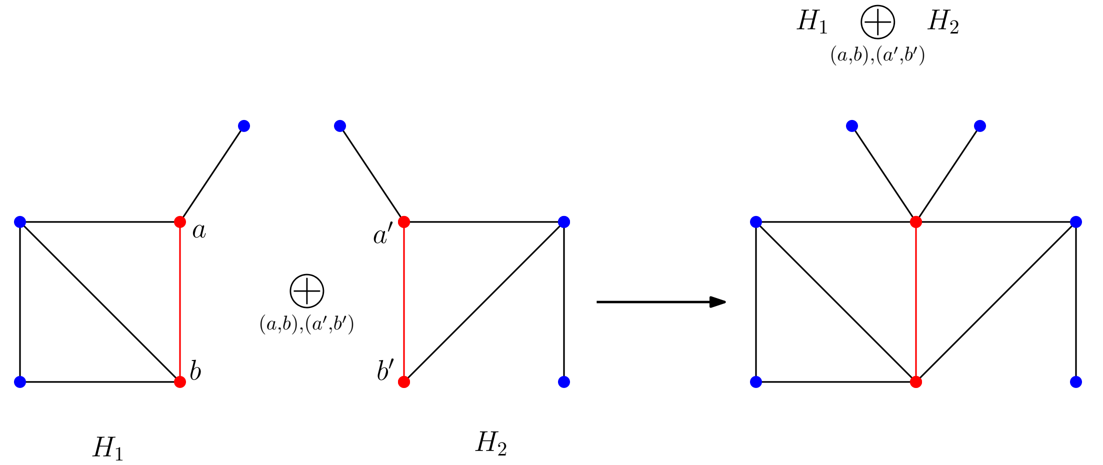

FIGURE 1. The $(a,b),(a',b')$-*vertex join* of the graphs H_1 and H_2.

⟨fig:vertexHab⟩

$\boldsymbol{W}$. The result depends on whether the number of colors $c_n = c$ is fixed or $c_n \to \infty$. We begin with fixed colors case.

1.2.1. *Asymptotic Distribution of* $\boldsymbol{\Gamma}(\mathcal{H}, \boldsymbol{G}_n)$ *for Fixed Number of Colors.* To describe the limiting distribution of $\boldsymbol{\Gamma}(\mathcal{H}, \boldsymbol{G}_n)$ we need a few notations. We begin by introducing the notion of the 2-point conditional kernel of a graphon W as follows:

Definition 1.2 (2-point conditional kernel)**.** Given a finite graph $H = (V(H), E(H))$ and a graphon W, define the *2-point conditional kernel of H with respect to W* as follows:[2]

$$W_H(x,y) := \sum_{1 \le a \ne b \le |V(H)|} \mathbb{E}\left(\prod_{(i,j) \in E(H)} W(U_i, U_j) \,\middle|\, U_a = x, U_b = y \right). \tag{1.11}$$

eq:WH

To describe the asymptotic variance of $\boldsymbol{\Gamma}(\mathcal{H}, \boldsymbol{G}_n)$ it is convenient to define the following graph operation.

definition:H2⟩? **Definition 1.3.** Suppose $H_1 = (V(H_1), E(H_1))$ and $H_2 = (V(H_2), E(H_2))$ be two graphs with vertex sets $V(H_1) = \{1, 2, \ldots, |V(H_1)|\}$ and $V(H_2) = \{1, 2, \ldots, |V(H_2)|\}$ and edge sets $E(H_1)$ and $E(H_2)$, respectively. Then for $1 \le a \ne b \le |V(H_1)|$ and $1 \le a' \ne b' \le |V(H_2)|$, the $(a,b),(a',b')$-*join* of H_1 and H_2 is the simple graph obtained by identifying the vertices a and b in H_1 with the vertices a' and b' in H_2, respectively. The resulting graph will be denoted by

$$H_1 \bigotimes_{(a,b),(a',b')} H_2.$$

An example is shown in Figure 1. (Notice that the $(a,b),(a',b')$-join of between 2 graphs can be different from the $(b,a),(a',b')$-join.)

With the above notations we can now state our result about the asymptotic distribution of $\boldsymbol{\Gamma}(\mathcal{H}, \boldsymbol{G}_n)$ for fixed number of colors. Throughout stochastic integrals are considered in the Weiner-Itô sense [31] (see also [32, Chapter 7]).

[2]Note that W_H can takes values greater than, hence, technically W_H it is not a graphon. However, $W_H \le |V(H)|(V(H)-1)$, hence, $W_H \in \mathscr{W}_1$, where $\mathscr{W}_1$ is the space of bounded, symmetric, measurable functions from $[0,1]^2 \to [0,\infty)$.

⟨thm:jointTHGn⟩ **Theorem 1.1.** *Suppose* $\{\boldsymbol{G_n}\}_{n\geq 1}$ *is a sequence of* d*-multiplexes converging to a* d*-graphon* $\boldsymbol{W}$*. Let* $\mathcal{H} = \{H_1, H_2, \ldots, H_d\}$ *be a collection of fixed graphs and* $\boldsymbol{\Gamma}(\mathcal{H}, \boldsymbol{G}_n)$ *be as defined in* (1.9)*. Assume, for* $1 \leq i \neq j \leq d$*, there exists* $\rho_{ij} \geq 0$ *such that*

$$\lim_{n\to\infty} \langle W_{H_i}^{G_n^{(i)}}, W_{H_j}^{G_n^{(j)}} \rangle = \rho_{ij}, \tag{1.12}$$

eq:WGnHcovariance

where $W_{H_i}^{G_n^{(i)}}$ *is the 2-point conditional kernel of the empirical graphon* $W^{G_n^{(i)}}$ *with respect to* H_i*, for* $1 \leq i \leq d$*. Then, as* $n \to \infty$*,*

$$\boldsymbol{\Gamma}(\mathcal{H}, \boldsymbol{G}_n) \xrightarrow{D} \sqrt{2\left(1 - \frac{1}{c}\right)} \boldsymbol{Z} + \frac{1}{\sqrt{c}} \sum_{a=1}^{c-1} \begin{pmatrix} \int_{[0,1]^2} (W_1)_{H_1}(x,y) \mathrm{d}B_x^{(a)} \mathrm{d}B_y^{(a)} \\ \vdots \\ \int_{[0,1]^2} (W_d)_{H_d}(x,y) \mathrm{d}B_x^{(a)} \mathrm{d}B_y^{(a)} \end{pmatrix},$$

where

- $(W_i)_{H_i}$ *is the 2-point conditional kernel of the graphon* W_i *with respect to* H_i*, for* $1 \leq i \leq d$*;*
- $\boldsymbol{Z} \sim N_d(0, \Sigma)$*, with* $\Sigma = ((\sigma_{ij}))_{1\leq i,j\leq d}$ *defined as:*

$$\sigma_{ij} := \begin{cases} \frac{1}{4} \sum\limits_{\substack{1\leq a\neq b\leq |V(H_i)| \\ 1\leq a'\neq b'\leq |V(H_i)|}} t\left(H_i \bigotimes_{(a,b),(a',b')} H_i, W_i\right) - \|(W_i)_{H_i}\|^2 & \text{for } i = j, \\ \rho_{ij} - \langle (W_i)_{H_i}, (W_j)_{H_j} \rangle & \text{for } i \neq j; \end{cases}$$

- $\{B_t^{(1)}, \ldots, B_t^{(c-1)}\}_{t\in[0,1]}$ *are i.i.d Brownian motions on* $[0,1]$ *which are independent of* $\boldsymbol{Z}$*.*

The proof of Theorem 1.1 is given in Section 3. A key ingredient of the proof is a general invariance principle, which shows the following: random multilinear forms indexed by the variables which encode the vertex coloring are asymptotically close in moments to multilinear forms indexed by appropriately chosen Gaussian variables (see Section 2). To apply this result, we first step express $T(H, G_n) - \mathbb{E}T(H, G_n)$, for any graph H, as a weighted sum of multilinear forms with degrees ranging from 2 to $|V(H)|$ (see Lemma 3.1). Next, we show all terms in the expansion with degree greater than 2 are asymptotically negligible (see Lemma 3.2). In other words, only the quadratic terms in the expansion of $T(H, G_n) - \mathbb{E}T(H, G_n)$ contributes to the limiting distribution. Then using the invariance principle the problem of deriving the joint distribution of $\boldsymbol{\Gamma}(\mathcal{H}, \boldsymbol{G}_n)$ reduces to finding the joint distribution of certain Gaussian quadratic forms, which is analyzed using the Cramér-Wold device and the spectral theorem.

Remark 1.1. Note that Theorem 1.1 assumes that the layers of the multiplex $\boldsymbol{G}_n$ converge jointly in the cut-distance and the pairwise overlaps of the 2-point conditional kernels have a limit. In Section 4 we provide examples which illustrate that both these assumptions are necessary for the limiting distribution of $\boldsymbol{\Gamma}(\mathcal{H}, \boldsymbol{G}_n)$ to exist.

One useful special case of Theorem 1.1 is when $H_1 = H_2 = \cdots = H_d = K_2$ is an edge. In this case, the 2-conditional kernel $W_{K_2}(x,y) = 2W(x,y)$ (recall (1.11)) and, for $1 \leq i \neq j \leq d$,

$$\langle W_{K_2}^{G_n^{(i)}}, W_{K_2}^{G_n^{(j)}} \rangle = \frac{2|E(G_n^{(i)}) \cap E(G_n^{(j)})|}{n^2},$$

where $E(G_n^{(i)})$ is the edge set of the graph $G_n^{(i)}$, for $1 \leq i \leq d$. Hence, Theorem 1.1 implies the following result:

⟨cor:K2joint⟩ **Corollary 1.2.** *Suppose $\{\boldsymbol{G_n}\}_{n\geq 1}$ is a sequence of d-multiplexes converging to a d-graphon $\boldsymbol{W}$. Also, assume that there exists $\rho_{ij} \geq 0$ such that*

$$\frac{2|E(G_n^{(i)}) \cap E(G_n^{(j)})|}{n^2} \to \rho_{ij}, \tag{1.13}$$ eq:covarianceed

for $1 \leq i \neq j \leq d$. Then, as $n \to \infty$,

$$\frac{1}{n}\left[\begin{pmatrix} T(K_2, G_n^{(1)}) \\ \vdots \\ T(K_2, G_n^{(d)}) \end{pmatrix} - \frac{1}{c}\begin{pmatrix} |E(G_n^{(1)})| \\ \vdots \\ |E(G_n^{(d)})| \end{pmatrix}\right] \xrightarrow{D} \frac{\sqrt{c-1}}{\sqrt{2}c}\boldsymbol{Z} + \frac{1}{2c}\sum_{a=1}^{c-1}\begin{pmatrix} \int_{[0,1]^2} W_1 \mathrm{d}B_x^{(a)}\mathrm{d}B_y^{(a)} \\ \vdots \\ \int_{[0,1]^2} W_d \mathrm{d}B_x^{(a)}\mathrm{d}B_y^{(a)} \end{pmatrix},$$

where

- *$\boldsymbol{Z} \sim N_d(0, \Sigma)$, with $\Sigma = ((\sigma_{ij}))_{1\leq i,j\leq d}$ defined as:*

$$\sigma_{ij} := \begin{cases} \int_{[0,1]^2} W_i(x,y)(1 - W_i(x,y))\mathrm{d}x\mathrm{d}y & \text{for } i = j, \\ \rho_{ij} - \int_{[0,1]^2} W_i(x,y)W_j(x,y)\mathrm{d}x\mathrm{d}y & \text{for } i \neq j; \end{cases}$$

- *$\{B_t^{(1)}, \ldots, B_t^{(c-1)}\}_{t\in[0,1]}$ are i.i.d Brownian motions on $[0,1]$ which are independent of $\boldsymbol{Z}$.*

If all the d layers of $\boldsymbol{G}_n$ are the same, that is, $G_n^{(1)} = G_n^{(2)} = \cdots = G_n^{(d)} = G_n$, then Theorem 1.1 gives joint distribution of the $(\Gamma(H_1, G_n), \ldots, \Gamma(H_d, G_n))$ for any collection of fixed graphs $\mathcal{H} = \{H_1, H_2, \ldots, H_d\}$. In this case, the limit of $\langle W_{H_i}^{G_n}, W_{H_j}^{G_n}\rangle$, for $1 \leq i \neq j \leq d$, can be derived from the convergence of $\{G_n\}_{n\geq 1}$ to a graphon W. Specifically, we have (see Lemma 3.5 for the proof),

$$\lim_{n\to\infty}\langle W_{H_1}^{G_n}, W_{H_2}^{G_n}\rangle = \frac{1}{4}\sum_{\substack{1\leq u\neq v\leq |V(H_1)| \\ 1\leq u'\neq v'\leq |V(H_2)|}} t\left(H_1 \bigotimes_{(u,v),(u',v')} H_2, W\right).$$

The above identity combined with Theorem 1.1 gives the following result:

⟨cor:jointTHGnW⟩ **Corollary 1.3.** *Suppose $\{G_n\}_{n\geq 1}$ is a sequence of graphs converging to a graphon W and let $\mathcal{H} = \{H_1, H_2, \ldots, H_d\}$ be a collection of fixed graphs. Then, as $n \to \infty$,*

$$\begin{pmatrix} \Gamma(H_1, G_n) \\ \vdots \\ \Gamma(H_d, G_n) \end{pmatrix} \xrightarrow{D} \sqrt{2\left(1 - \frac{1}{c}\right)}\boldsymbol{Z} + \frac{1}{\sqrt{c}}\sum_{a=1}^{c-1}\begin{pmatrix} \int_{[0,1]^2} W_{H_1}(x,y)\mathrm{d}B_x^{(a)}\mathrm{d}B_y^{(a)} \\ \vdots \\ \int_{[0,1]^2} W_{H_d}(x,y)\mathrm{d}B_x^{(a)}\mathrm{d}B_y^{(a)} \end{pmatrix},$$

where

- $\boldsymbol{Z} \sim N_d(0, \Sigma)$, *with* $\Sigma = ((\sigma_{ij}))_{1 \le i,j \le d}$ *defined as:*

$$\sigma_{ij} := \begin{cases} \frac{1}{4} \sum_{\substack{1 \le a \ne b \le |V(H_i)| \\ 1 \le a' \ne b' \le |V(H_i)|}} t\left(H_i \bigotimes_{(a,b),(a',b')} H_i, W\right) - \|W_{H_i}\|^2 & \text{for } i = j, \\ \frac{1}{4} \sum_{\substack{1 \le a \ne b \le |V(H_i)| \\ 1 \le a' \ne b' \le |V(H_j)|}} t\left(H_i \bigotimes_{(a,b),(a',b')} H_j, W\right) - \langle W_{H_i}, W_{H_j} \rangle & \text{for } i \ne j; \end{cases}$$

- $\{B_t^{(1)}, \ldots, B_t^{(c-1)}\}_{t \in [0,1]}$ *are i.i.d Brownian motions on* $[0,1]$ *which are independent of* $\boldsymbol{Z}$*.*

Note that by taking $\mathcal{H} = \{H\}$ is a singleton we recover from Corollary 1.3 the marginal distribution of $T(H, G_n)$, which was proved in [9, Theorem 3.1]. In this case the limiting distribution can be alternately expressed in terms of the eigenvalues of the 2-point conditional kernel (recall (1.11)) as discussed in the following remark.

Remark 1.2. Note that by the spectral theorem, for any graph H, the 2-point conditional kernel W_H can be expressed as (see [38, Section 7.5]):

$$W_H(x,y) = \sum_{s=1}^{\infty} \lambda_s \phi_s(x) \phi_s(y),$$

where $\{\lambda_s\}_{s \ge 1}$ are the eigenvalues and $\{\phi_s\}_{s \ge 1}$ are an orthonormal collection of eigenvectors of the operator $\mathcal{T}_{W_H}(f) = \int_0^1 W(x,y) f(y) \mathrm{d}y$. Hence, for each $a \in [c]$, by the linearity of stochastic integrals and the product formula (see [32, Page 100]):

$$\begin{aligned} \int_{[0,1]^2} W_H(x,y) \mathrm{d}B_x^{(a)} \mathrm{d}B_y^{(a)} &= \sum_{s=1}^{\infty} \lambda_s \int_{[0,1]^2} \phi_s(x)\phi_s(y) \mathrm{d}B_x^{(a)} \mathrm{d}B_y^{(a)} \\ &= \sum_{s=1}^{\infty} \lambda_s \left(\left(\int_0^1 \phi_s(x) \mathrm{d}B_x \right)^2 - \int_0^1 \phi_s(x)^2 \mathrm{d}x \right) \\ &= \sum_{s=1}^{\infty} \lambda_s \left(\left(\int_0^1 \phi_s(x) \mathrm{d}B_x^{(a)} \right)^2 - 1 \right) \qquad \text{(by orthonormality)} \\ &\stackrel{D}{=} \sum_{s=1}^{\infty} \lambda_s \left(Z_{s,a}^2 - 1 \right), \end{aligned}$$

where $Z_{s,a} \stackrel{D}{=} \int_0^1 \phi_s(x) \mathrm{d}B_x$. Note that by orthonormality $\{Z_{s,a}\}_{s \ge 1, a \in [c]}$ is a collection of i.i.d. $N(0,1)$ random variables. Hence, from Corollary 1.3 we obtain the following alternative expression of the limiting distribution of $\Gamma(H, G_n)$:

$$\begin{aligned} \Gamma(H, G_n) &\xrightarrow{D} \sqrt{2\left(1 - \frac{1}{c}\right)} Z + \frac{1}{\sqrt{c}} \sum_{a=1}^{c-1} \sum_{s=1}^{\infty} \lambda_s \left(Z_{s,a}^2 - 1 \right) \\ &\stackrel{D}{=} \sqrt{2\left(1 - \frac{1}{c}\right)} Z + \frac{1}{\sqrt{c}} \sum_{s=1}^{\infty} \lambda_s \xi_s, \end{aligned}$$

where $\{\xi_s\}_{s\geq 1}$ is a collection of i.i.d. $\chi^2_{c-1} - (c-1)$ random variables which is independent of $Z \sim N(0, \sigma^2)$, with

$$\sigma^2 = \frac{1}{4} \sum_{\substack{1 \leq a \neq b \leq |V(H)| \\ 1 \leq a' \neq b' \leq |V(H)|}} t\left(H_i \bigotimes_{(a,b),(a',b')} H, W \right) - \|W_H\|^2.$$

This recovers the result in [9, Theorem 1.3].

2. A General Invariance Principle

⟨sec:invariance⟩ For each $v \in [n]$ and $a \in [c]$, let

$$\tilde{X}_{v,a} = \sqrt{c}\left(\mathbf{1}\{X_v = a\} - \frac{1}{c} \right). \tag{2.1}$$

?eq:Xva?

Let $\{Z_{v,a}\}_{v\in[n],a\in[c]}$ be a collection of i.i.d $N(0,1)$ and define

$$\tilde{Z}_{v,a} := Z_{v,a} - \frac{1}{c}\sum_{a=1}^{c} Z_{v,a}. \tag{2.2}$$

eq:Zva

We collect some easy facts about the random variables $\{\tilde{X}_{v,a}\}_{v\in[n],a\in[c]}$ and $\{\tilde{Z}_{v,a}\}_{v\in[n],a\in[c]}$ in the following observation:

⟨obs:XZva⟩ **Observation 2.1.** *Let $\{\tilde{X}_{v,a}\}_{v\in[n],a\in[c]}$ and $\{\tilde{Z}_{v,a}\}_{v\in[n],a\in[c]}$ be as defined above. Then the following hold:*

ments_match:1⟩? (a) $\mathbb{E}\tilde{X}_{v,a} = \mathbb{E}\tilde{Z}_{v,a} = 0$, *for* $v \in [n], a \in [c]$.

ments_match:2⟩? (b) $\mathrm{Var}(\tilde{X}_{v,a}) = \mathrm{Var}(\tilde{Z}_{v,a}) = 1 - \frac{1}{c}$, *for* $v \in [n], a \in [c]$.

ments_match:3⟩? (c) $\mathrm{Cov}(\tilde{X}_{v,a}, \tilde{X}_{v,b}) = \mathrm{Cov}(\tilde{Z}_{v,a}, \tilde{Z}_{v,b}) = -\frac{1}{c}$, *for* $v \in [n]$ *and* $a \neq b \in [c]$.

ndependenceuv⟩? (d) $\{\tilde{X}_{u,a}\}_{a\in[c]} \perp \{\tilde{X}_{v,a}\}_{a\in[c]}$ *and* $\{\tilde{Z}_{u,a}\}_{a\in[c]} \perp \{\tilde{Z}_{v,a}\}_{a\in[c]}$, *for* $u \neq v \in [n]$.

Hereafter, denote

$$\boldsymbol{X} = (X_{v,a})_{v\in[n],a\in[c]},\ \tilde{\boldsymbol{X}} = (\tilde{X}_{v,a})_{v\in[n],a\in[c]},\ \boldsymbol{Z} = (Z_{v,a})_{v\in[n],a\in[c]},\ \text{and}\ \tilde{\boldsymbol{Z}} = (\tilde{Z}_{v,a})_{v\in[n],a\in[c]}.$$

Fix an integer $r \geq 1$. Then for a function $f : [n]_r \to \mathbb{R}$, define

$$\mathcal{T}(f; \tilde{\boldsymbol{X}}) := \frac{1}{n^{\frac{r}{2}}\sqrt{c}} \sum_{a=1}^{c} \sum_{\boldsymbol{s}\in[n]_r} f(\boldsymbol{s}) \prod_{j=1}^{r} \tilde{X}_{s_j,a}. \tag{2.3}$$

eq:Tf

Define $\mathcal{T}(f; \boldsymbol{X})$ and $\mathcal{T}(f; \tilde{\boldsymbol{Z}})$ similarly. Let $\mathcal{S}_r$ denote the set of all permutations of $[r]$. Then define the symmetrization of f as follows:

$$\tilde{f}(s_1, \cdots s_r) = \frac{1}{r!} \sum_{\sigma\in\mathcal{S}_r} f(s_{\sigma(1)}, \cdots, s_{\sigma(r)}).$$

For functions $f, f' : [n]_r \to \mathbb{R}$, their inner product is defined as:

$$\langle f, f' \rangle := \frac{1}{n^r} \sum_{\boldsymbol{s}\in[n]_r} f(\boldsymbol{s}) f'(\boldsymbol{s}).$$

and let $\|f\| := \sqrt{\langle f, f\rangle}$ be the associated norm. For a $\boldsymbol{s} \in [n]_r$, let $\underline{\boldsymbol{s}}$ denote the (unordered) set formed by the entries of $\boldsymbol{s}$. (For example, if $\boldsymbol{s} = (4, 2, 5)$, then $\underline{\boldsymbol{s}} = \{2, 4, 5\}$.) Notice that

$$\langle \tilde{f}, \tilde{f}'\rangle = \langle \tilde{f}, f'\rangle = \langle f, \tilde{f}'\rangle = \frac{1}{n^r}\frac{1}{r!}\sum_{\substack{\boldsymbol{s},\boldsymbol{s}'\in[n]_r \\ \underline{\boldsymbol{s}}=\underline{\boldsymbol{s}}'}} f(\boldsymbol{s})f'(\boldsymbol{s}'). \tag{2.4}$$ eq:ip_sym_S_s

We can then define the following bilinear operation:

$$\langle f, f'\rangle_\bullet := \langle f, \tilde{f}'\rangle$$

and also the associated pseudo-norm $\|f\|_\bullet := \sqrt{\langle f, f\rangle_\bullet} = \|\tilde{f}\|$.

⟨lm:varianceTf⟩ **Lemma 2.1.** *Fix integers $r, r' \geq 1$. Then for functions $f : [n]_r \to \mathbb{R}$ and $f' : [n]_{r'} \to \mathbb{R}$, the following hold:*

- *If $r \neq r'$, then*

$$\mathrm{Cov}\left(\mathcal{T}(f, \tilde{\boldsymbol{X}}), \mathcal{T}(f', \tilde{\boldsymbol{X}})\right) = 0. \tag{2.5}$$ eq:cov_T_f_zero

- *If $r = r'$, then*

$$\mathrm{Cov}\left(\mathcal{T}(f, \tilde{\boldsymbol{X}}), \mathcal{T}(f', \tilde{\boldsymbol{X}})\right) = \mathbb{E}\left(\mathcal{T}(f, \tilde{\boldsymbol{X}})\mathcal{T}(f', \tilde{\boldsymbol{X}})\right) = \eta \cdot r!\langle f, f'\rangle_\bullet,$$

  where $\eta := \left(1 - \frac{1}{c}\right)^r + (c-1)\left(-\frac{1}{c}\right)^r$.

The same holds for $\mathrm{Cov}(\mathcal{T}(f, \tilde{\boldsymbol{Z}}), \mathcal{T}(f', \tilde{\boldsymbol{Z}}))$.

Proof. Note that for any $\boldsymbol{s} \in [n]_r$ and $a \in [c]$, $\mathbb{E}(\prod_{j=1}^r \tilde{X}_{s_j,a}) = 0$. This implies, recall (2.3), $\mathbb{E}[\mathcal{T}(f, \tilde{\boldsymbol{X}})] = 0$. Hence,

$$\begin{aligned}\mathrm{Cov}\left(\mathcal{T}(f, \tilde{\boldsymbol{X}}), \mathcal{T}(f', \tilde{\boldsymbol{X}})\right) &= \mathbb{E}\left(\mathcal{T}(f, \tilde{\boldsymbol{X}})\mathcal{T}(f', \tilde{\boldsymbol{X}})\right) \\ &= \frac{1}{n^{\frac{r+r'}{2}}c}\sum_{a,a'\in[c]}\sum_{\boldsymbol{s}\in[n]_r \boldsymbol{s}'\in[n]_{r'}} f(\boldsymbol{s})f'(\boldsymbol{s}')\mathbb{E}\left(\prod_{j\in\underline{\boldsymbol{s}}}\tilde{X}_{s_j,a}\prod_{j\in\underline{\boldsymbol{s}}'}\tilde{X}_{s'_j,a'}\right),\end{aligned} \tag{2.6}$$ eq:covTf

where $\boldsymbol{s} = (s_1, s_2, \ldots, s_r)$ and $\boldsymbol{s}' = (s'_1, s'_2, \ldots, s'_{r'})$, for $\boldsymbol{s} \in [n]_r$ and $\boldsymbol{s}' \in [n]_{r'}$. Observe that if $\boldsymbol{s} \neq \boldsymbol{s}'$ (that is, either $\underline{\boldsymbol{s}}\backslash\underline{\boldsymbol{s}}'$ or $\underline{\boldsymbol{s}}'\backslash\underline{\boldsymbol{s}}$ is non-empty), then

$$\mathbb{E}\left(\prod_{j\in\underline{\boldsymbol{s}}}\tilde{X}_{s_j,a}\prod_{j\in\underline{\boldsymbol{s}}'}\tilde{X}_{s'_j,a'}\right) = 0.$$

To begin with, suppose $r \neq r'$. Then $\boldsymbol{s} \neq \boldsymbol{s}'$, for any $\boldsymbol{s} \in [n]_r$ and $\boldsymbol{s}' \in [n]_{r'}$. Hence, in this case all the terms in (2.6) are zero, and the result in (2.5) follows.

Now, suppose $r = r'$ and $\boldsymbol{s}, \boldsymbol{s}' \in [n]_r$ is such that $\underline{\boldsymbol{s}} = \underline{\boldsymbol{s}}'$. Then

$$\begin{aligned}\frac{1}{c}\sum_{a,a'\in[c]}\mathbb{E}\left(\prod_{j\in\underline{\boldsymbol{s}}}\tilde{X}_{s_j,a}\prod_{j\in\underline{\boldsymbol{s}}'}\tilde{X}_{s_j,a'}\right) &= \frac{1}{c}\sum_{a,a'\in[c]}\mathbb{E}\left(\prod_{j\in\underline{\boldsymbol{s}}}\tilde{X}_{s_j,a}\tilde{X}_{s_j,a'}\right) \\ &= \frac{1}{c}\sum_{a\in[c]}\mathbb{E}\left(\prod_{j\in\underline{\boldsymbol{s}}}\tilde{X}^2_{s_j,a}\right) + \frac{1}{c}\sum_{a\neq a'\in[c]}\mathbb{E}\left(\prod_{j\in\underline{\boldsymbol{s}}}\tilde{X}_{s_j,a}\tilde{X}_{s_j,a'}\right) \\ &= \left(1 - \frac{1}{c}\right)^r + (c-1)\left(-\frac{1}{c}\right)^r = \eta.\end{aligned}$$

Hence, from (2.6),

$$\mathrm{Cov}\left(\mathcal{T}(f,\tilde{\boldsymbol{X}}),\mathcal{T}(f',\tilde{\boldsymbol{X}})\right)=\frac{1}{n^r c}\sum_{a,a'\in[c]}\sum_{\substack{\boldsymbol{s}\in[n]_r,\boldsymbol{s}'\in[n]_r\\ \underline{\boldsymbol{s}}=\underline{\boldsymbol{s}}'}} f(\boldsymbol{s})f'(\boldsymbol{s}')\mathbb{E}\left(\prod_{j\in\underline{\boldsymbol{s}}}\tilde{X}_{s_j,a}\prod_{j\in\underline{\boldsymbol{s}}'}\tilde{X}_{s_j,a'}\right)$$
$$=\eta\cdot\frac{1}{n^r}\sum_{\substack{\boldsymbol{s}\in[n]_r,\boldsymbol{s}'\in[n]_{r'}\\ \underline{\boldsymbol{s}}=\underline{\boldsymbol{s}}'}} f(\boldsymbol{s})f'(\boldsymbol{s}')$$
$$=\eta\cdot r!\langle f,f'\rangle_{\bullet},$$

where the last step uses (2.4). □

Next, we show that $\mathcal{T}(\cdot;\tilde{\boldsymbol{X}})$ satisfies an invariance principle. Specifically, we show that the moments of $\mathcal{T}(\cdot;\tilde{\boldsymbol{X}})$ and $\mathcal{T}(\cdot;\tilde{\boldsymbol{Z}})$ are asymptotically close, for any finite collection of bounded functions.

ppn:mthd_mmnts⟩ **Proposition 2.1.** *Fix $K\geq 1$ and integers $r_1,r_2,\ldots,r_K\geq 1$. For $1\leq k\leq K$, let $\{f_n^{(k)}\}_{n\geq 1}$ be sequence of functions such that $f_n^{(k)}:[n]_{r_k}\to[-M,M]$, for some $M>0$. Then*

$$\lim_{n\to\infty}\left|\mathbb{E}\prod_{k=1}^{K}\mathcal{T}\left(f_n^{(k)};\tilde{\boldsymbol{X}}\right)-\mathbb{E}\prod_{k=1}^{K}\mathcal{T}\left(f_n^{(k)};\tilde{\boldsymbol{Z}}\right)\right|=0. \tag{2.7}$$?eq:mthd_mmnts_

Proof. For notational convenience denote,

$$\Delta_{n,K}:=\left|\mathbb{E}\prod_{k=1}^{K}\mathcal{T}\left(f_n^{(k)};\tilde{\boldsymbol{X}}\right)-\mathbb{E}\prod_{k=1}^{K}\mathcal{T}\left(f_n^{(k)};\tilde{\boldsymbol{Z}}\right)\right|.$$

Note that

$$\Delta_{n,K}=\frac{1}{n^{\frac{r_\bullet}{2}}c^{\frac{K}{2}}}\left|\sum_{\boldsymbol{a}\in[c]^K}\sum_{\substack{\boldsymbol{s}^{(1)},\ldots,\boldsymbol{s}^{(K)}\\ \boldsymbol{s}^{(k)}\in[n]_{r_k}}}\prod_{k=1}^{K}f_n^{(k)}(\boldsymbol{s}^{(k)})\left\{\mathbb{E}\left(\prod_{k=1}^{K}\prod_{j\in[n]_{r_k}}\tilde{X}_{s_j^{(k)},a_k}\right)-\mathbb{E}\left(\prod_{k=1}^{K}\prod_{j\in[n]_{r_k}}\tilde{Z}_{s_j^{(k)},a_k}\right)\right\}\right|,$$

where $r_\bullet=\sum_{k=1}^K r_k$, $\boldsymbol{a}=(a_1,a_2,\ldots,a_K)\in[c]^K$ and $\boldsymbol{s}^{(k)}=(s_1^{(k)},s_2^{(k)},\ldots,s_{r_k}^{(k)})\in[n]_{r_K}$, for $1\leq k\leq K$. Using the fact that $|f_n^{(k)}|\leq M$, for $1\leq k\leq K$, gives

$$\Delta_{n,K}\leq\frac{M^K}{n^{\frac{1}{2}\sum_{k=1}^K r_k}c^{\frac{K}{2}}}\sum_{\boldsymbol{a}\in[c]^K}\sum_{\substack{\boldsymbol{s}^{(1)},\ldots,\boldsymbol{s}^{(K)}\\ \boldsymbol{s}^{(k)}\in[n]_{r_k}}}\left|\mathbb{E}\left(\prod_{k=1}^{K}\prod_{j\in[n]_{r_k}}\tilde{X}_{s_j^{(k)},a_k}\right)-\mathbb{E}\left(\prod_{k=1}^{K}\prod_{j\in[n]_{r_k}}\tilde{Z}_{s_j^{(k)},a_k}\right)\right|. \tag{2.8}$$ eq:mthd_mmnts_c

Given $\boldsymbol{a}\in[c]^K$ and $\boldsymbol{S}:=(\boldsymbol{s}^{(1)},\cdots,\boldsymbol{s}^{(K)})$, such that $\boldsymbol{s}^{(k)}\in[n]_{r_k}$, for $1\leq k\leq K$, define the edge-colored hypergraph $F_{\boldsymbol{S},\boldsymbol{a}}=(V(F_{\boldsymbol{S},\boldsymbol{a}}),E(F_{\boldsymbol{S},\boldsymbol{a}}))$ as follows:

$$V(F_{\boldsymbol{S},\boldsymbol{a}}):=\bigcup_{k=1}^{K}\underline{\boldsymbol{s}}^{(k)}\quad\text{and}\quad E(F_{\boldsymbol{S_K},\boldsymbol{a}}):=\{\underline{\boldsymbol{s}}^{(1)},\ldots,\underline{\boldsymbol{s}}^{(K)}\},$$

and the color of the hyperedge $\underline{\boldsymbol{s}}^{(k)}$ is a_k, for $1 \le k \le K$. Also, define

$$\tilde{\boldsymbol{X}}_{F_{\boldsymbol{S},\boldsymbol{a}}} := \prod_{k=1}^{K} \prod_{j \in [n]_{r_k}} \tilde{X}_{s_j^{(k)}, a_k} \text{ and } \tilde{\boldsymbol{Z}}_{F_{\boldsymbol{S},\boldsymbol{a}}} := \prod_{k=1}^{K} \prod_{j \in [n]_{r_k}} \tilde{Z}_{s_j^{(k)}, a_k}.$$

The following lemma records 2 crucial properties of $\mathbb{E}\tilde{\boldsymbol{X}}_{F_{\boldsymbol{S},\boldsymbol{a}}}$ and $\mathbb{E}\tilde{\boldsymbol{Z}}_{F_{\boldsymbol{S},\boldsymbol{a}}}$.

⟨lm:XZ⟩ **Lemma 2.2.** *For $j \in V(F_{\boldsymbol{S},\boldsymbol{a}})$, denote by d_j the degree of vertex $j \in V(F_{\boldsymbol{S},\boldsymbol{a}})$. Then the following hold:*

- *If there exists $j \in V(F_{\boldsymbol{S},\boldsymbol{a}})$ such that $d_j = 1$, then*
$$\mathbb{E}\tilde{\boldsymbol{X}}_{F_{\boldsymbol{S},\boldsymbol{a}}} = \mathbb{E}\tilde{\boldsymbol{Z}}_{F_{\boldsymbol{S},\boldsymbol{a}}} = 0.$$
- *If $d_j = 2$, for all $j \in V(F_{\boldsymbol{S},\boldsymbol{a}})$, then*
$$\mathbb{E}\tilde{\boldsymbol{X}}_{F_{\boldsymbol{S},\boldsymbol{a}}} = \mathbb{E}\tilde{\boldsymbol{Z}}_{F_{\boldsymbol{S},\boldsymbol{a}}}.$$

Proof. Note that

$$\mathbb{E}\tilde{\boldsymbol{X}}_{F_{\boldsymbol{S},\boldsymbol{a}}} = \mathbb{E}\left(\prod_{j \in V(F_{\boldsymbol{S},\boldsymbol{a}})} \prod_{a=1}^{c} \tilde{X}_{j,a}^{\ell_{j,a}} \right) = \prod_{j \in V(F_{\boldsymbol{S},\boldsymbol{a}})} \mathbb{E}\left(\prod_{a=1}^{c} \tilde{X}_{j,a}^{\ell_{j,a}} \right), \tag{2.9}$$ eq:XFs

where $\ell_{j,a}$ be the number of hyperedges with color $a \in [c]$ containing the vertex $j \in V(F_{\boldsymbol{S},\boldsymbol{a}})$. Now, if $j \in V(F_{\boldsymbol{S},\boldsymbol{a}})$ is such that $d_j = 1$, then there exists $b \in [c]$ such that $\ell_{j,b} = 1$ and $\ell_{j,a} = 0$, for all $a \in [c] \backslash b$. This means,

$$\mathbb{E}\left(\prod_{a=1}^{c} \tilde{X}_{j,a}^{\ell_{j,a}} \right) = \mathbb{E}\tilde{X}_{j,b} = 0$$

and, consequently, $\mathbb{E}\tilde{\boldsymbol{X}}_{F_{\boldsymbol{S},\boldsymbol{a}}} = 0$. By the same argument we also have, $\mathbb{E}\tilde{\boldsymbol{Z}}_{F_{\boldsymbol{S},\boldsymbol{a}}} = 0$.

Next, suppose $d_j = 2$, for all $j \in V(F_{\boldsymbol{S},\boldsymbol{a}})$. Then there are 2 possibilities:

- There exists $b \in [c]$ such that $\ell_{j,b} = 2$ and $\ell_{j,a} = 0$, for all $a \in [c] \backslash \{b\}$. This means,
$$\mathbb{E}\left(\prod_{a=1}^{c} \tilde{X}_{j,a}^{\ell_{j,a}} \right) = \mathbb{E}\tilde{X}_{j,b}^2 = 1 - \frac{1}{c},$$
by Observation 2.1. By the same argument,
$$\mathbb{E}\left(\prod_{a=1}^{c} \tilde{Z}_{j,a}^{\ell_{j,a}} \right) = \mathbb{E}\tilde{Z}_{j,b}^2 = 1 - \frac{1}{c}.$$
- There exists $b \ne b' \in [c]$ such that $\ell_{j,b} = \ell_{j,b'} = 1$ and $\ell_{j,a} = 0$, for all $a \in [c] \backslash \{b, b'\}$. This means,
$$\mathbb{E}\left(\prod_{a=1}^{c} \tilde{X}_{j,a}^{\ell_{j,a}} \right) = \mathbb{E}\tilde{X}_{j,b}\tilde{X}_{j,b'} = -\frac{1}{c},$$
by Observation 2.1. By the same argument,
$$\mathbb{E}\left(\prod_{a=1}^{c} \tilde{Z}_{j,a}^{\ell_{j,a}} \right) = \mathbb{E}\tilde{Z}_{j,b}\tilde{Z}_{j,b'} = -\frac{1}{c}.$$

Hence, if $d_j = 2$, then

$$\mathbb{E}\left(\prod_{a=1}^{c} \tilde{X}_{j,a}^{\ell_{j,a}}\right) = \mathbb{E}\left(\prod_{a=1}^{c} \tilde{Z}_{j,a}^{\ell_{j,a}}\right)$$

Hence, taking product over $j \in V(F_{\boldsymbol{S},\boldsymbol{a}})$ (recall (2.9)), gives $\mathbb{E}\tilde{\boldsymbol{X}}_{F_{\boldsymbol{S},\boldsymbol{a}}} = \mathbb{E}\tilde{\boldsymbol{Z}}_{F_{\boldsymbol{S},\boldsymbol{a}}}$. □

The next lemma gives bounds on $\mathbb{E}\tilde{\boldsymbol{X}}_{F_{\boldsymbol{S},\boldsymbol{a}}}$ and $\mathbb{E}\tilde{\boldsymbol{Z}}_{F_{\boldsymbol{S},\boldsymbol{a}}}$.

ctationXZbound⟩ **Lemma 2.3.** *For any* $\boldsymbol{a} = (a_1, a_2, \ldots, a_K) \in [c]^K$ *and* $\boldsymbol{S} = (\boldsymbol{s}^{(1)}, \cdots, \boldsymbol{s}^{(K)})$, *with* $\boldsymbol{s}^{(k)} \in [n]_{r_k}$, *for* $1 \leq k \leq K$, *we have*

$$\mathbb{E}\tilde{\boldsymbol{X}}_{F_{\boldsymbol{S},\boldsymbol{a}}} \lesssim_{K,c} 1, \tag{2.10}$$ eq:mthd_mmnts_p

where $\nu(F_{\boldsymbol{S},\boldsymbol{a}})$ *is the number of connected components of* $F_{\boldsymbol{S},\boldsymbol{a}}$. *Moreover,* $\mathbb{E}\tilde{\boldsymbol{Z}}_{F_{\boldsymbol{S},\boldsymbol{a}}} \lesssim_{r_1,\ldots,r_K} 1$.

Proof. To begin with, define $\mathcal{C}_j := \{a \in [c] : \ell_{j,a} \geq 1\}$, for $j \in V(F_{\boldsymbol{S},\boldsymbol{a}})$. Then, for each $j \in V(F_{\boldsymbol{S},\boldsymbol{a}})$,

$$\begin{aligned}
\left|\mathbb{E}\left(\prod_{a=1}^{c} \tilde{X}_{j,a}^{\ell_{j,a}}\right)\right| &= \left|\mathbb{E}\left(\prod_{a\in\mathcal{C}_j} \tilde{X}_{j,a}^{\ell_{j,a}}\right)\right| \\
&= c^{\frac{1}{2}\sum_{a\in\mathcal{C}_j}\ell_{j,a}} \left|\mathbb{E}\prod_{a\in\mathcal{C}_j}\left(\mathbf{1}\{X_j = a\} - \frac{1}{c}\right)^{\ell_{j,a}}\right| \\
&\leq c^{\frac{d_j}{2}}\mathbb{E}\prod_{a\in\mathcal{C}_j}\left|\mathbf{1}\{X_j = a\} - \frac{1}{c}\right|^{\ell_{j,a}} \\
&= c^{\frac{d_j}{2}}\sum_{b=1}^{c}\mathbb{P}(X_j = b)\left(1 - \frac{1}{c}\right)^{\ell_{j,b}}\prod_{a\in\mathcal{C}_j\setminus\{b\}}\frac{1}{c^{\ell_{j,a}}} \\
&\leq c^{\frac{d_j}{2}-1}\sum_{b=1}^{c} c^{-\sum_{a\in\mathcal{C}_j\setminus\{b\}}\ell_{j,a}} \\
&\leq c^{\frac{d_j}{2}-1}\sum_{b\in\mathcal{C}_j} c^{-\sum_{a\in\mathcal{C}_j\setminus\{b\}}\ell_{j,a}} + c^{\frac{d_j}{2}-1}\sum_{b\notin\mathcal{C}_j} c^{-\sum_{a\in\mathcal{C}_j}\ell_{j,a}} \qquad (2.11) \\
&\leq |\mathcal{C}_j| c^{\frac{d_j}{2}-|\mathcal{C}_j|} + c^{\frac{d_j}{2}-|\mathcal{C}_j|} \lesssim_K c^{\frac{d_j}{2}-|\mathcal{C}_j|}, \qquad (2.12)
\end{aligned}$$

eq:TC eq:TK

where (2.11) uses $\ell_{j,a} \geq 1$, for $a \in \mathcal{C}_j$, and (2.12) uses $|\mathcal{C}_j| \leq K$, for $j \in V(F_{\boldsymbol{S},\boldsymbol{a}})$. Hence, from (2.9),

$$\left|\mathbb{E}\tilde{\boldsymbol{X}}_{\boldsymbol{S},\boldsymbol{a}}\right| \lesssim c^{\frac{1}{2}\sum_{j\in V(F_{\boldsymbol{S},\boldsymbol{a}})} d_j - \sum_{j\in V(F_{\boldsymbol{S},\boldsymbol{a}})}|\mathcal{C}_j|} = c^{\frac{1}{2}\sum_{k\in[K]} r_k - \sum_{j\in V(F_{\boldsymbol{S},\boldsymbol{a}})}(|\mathcal{C}_j|-1) - |V(F_{\boldsymbol{S},\boldsymbol{a}})|} \lesssim_{K,c} 1.$$

This proves (2.10).

Next, we bound $\mathbb{E}\tilde{\boldsymbol{Z}}_{F_{\boldsymbol{S},\boldsymbol{a}}}$. First, observe that for each $j \in V(F_{\boldsymbol{S},\boldsymbol{a}})$,

$$\begin{aligned}
\left|\mathbb{E}\left(\prod_{a=1}^{c} \tilde{Z}_{j,a}^{\ell_{j,a}}\right)\right| &\leq \mathbb{E}\prod_{a\in\mathcal{C}_j}\left|Z_{j,a} - \frac{1}{c}\sum_{a=1}^{c} Z_{j,a}\right|^{\ell_{j,a}} \\
&\leq \prod_{a\in\mathcal{C}_j}\left(\mathbb{E}\left|Z_{j,a} - \frac{1}{c}\sum_{a=1}^{c} Z_{j,a}\right|^{\ell_{j,a}|\mathcal{C}_j|}\right)^{\frac{1}{\mathcal{C}_j}} \qquad \text{(by Hölder's inequality)}
\end{aligned}$$

$$\leq \prod_{a \in \mathcal{C}_j} \left(\left(1 - \frac{1}{c}\right)^{\frac{\ell_{j,a}|\mathcal{C}_j|}{2}} \mathbb{E}\,|N(0,1)|^{\ell_{j,a}|\mathcal{C}_j|} \right)^{\frac{1}{\mathcal{C}_j}}$$

$$\leq \prod_{a \in \mathcal{C}_j} \left(\mathbb{E}\,|N(0,1)|^{\ell_{j,a}|\mathcal{C}_j|} \right)^{\frac{1}{\mathcal{C}_j}} \lesssim_K 1,$$

since $\ell_{j,a} \leq K$ and $|\mathcal{C}_j| \leq K$, for $j \in V(F_{\boldsymbol{S},\boldsymbol{a}})$. This implies,

$$|\mathbb{E}\tilde{\boldsymbol{Z}}_{F_{\boldsymbol{S},\boldsymbol{a}}}| = \left| \mathbb{E}\left(\prod_{j \in V(F_{\boldsymbol{S},\boldsymbol{a}})} \prod_{a=1}^{c} \tilde{Z}_{j,a}^{\ell_{j,a}} \right) \right| = \prod_{j \in V(F_{\boldsymbol{S},\boldsymbol{a}})} \left| \mathbb{E}\left(\prod_{a=1}^{c} \tilde{Z}_{j,a}^{\ell_{j,a}} \right) \right| \lesssim_{r_1, r_2, \ldots, r_K} 1,$$

since $|V(F_{\boldsymbol{S},\boldsymbol{a}})| \leq \sum_{k=1}^{K} r_k$. This completes the proof of Lemma 2.3. □

Now, let $\mathcal{F}$ be the set of colored hypergraphs with the following properties:

- $F \in \mathcal{F}$ is a hypergraph with most $\sum_{k=1}^{K} r_k$ vertices and at most K hyperedges and every hyperedge of F is assigned a color from a set of size at most K.
- all vertices of F have degree at least two, and
- some vertex of F has degree at least 3.

Note that for $F \in \mathcal{F}$ there are $O(n^{|V(F)|})$ terms in the joint sum (2.8) such that $F_{\boldsymbol{S},\boldsymbol{a}}$ is isomorphic to F. This implies, from (2.8) and Lemma 2.2,

$$\Delta_{n,K} \lesssim_{M,K,c} \frac{1}{n^{\frac{1}{2}\sum_{k=1}^{K} r_k}} \sum_{F \in \mathcal{F}} n^{|V(F)|} |\mathbb{E}\tilde{\boldsymbol{X}}_{F_{\boldsymbol{S},\boldsymbol{a}}} - \mathbb{E}\tilde{\boldsymbol{Z}}_{F_{\boldsymbol{S},\boldsymbol{a}}}|$$

$$\lesssim_{M,K,c} \sum_{F \in \mathcal{F}} n^{|V(F)| - \frac{1}{2}\sum_{k=1}^{K} r_k}. \tag{2.13}$$ eq:mthd_mmnts_s

Note that $\sum_{k=1}^{K} r_k = \sum_{e \in E(F)} |e|$. Also, for $F \in \mathcal{F}$, we have that $d_j \geq 2$ for all $j \in V(F)$ with strict inequality for some j. Therefore,

$$\frac{1}{2}\sum_{k=1}^{K} r_k = \frac{1}{2}\sum_{e \in E(F)} |e| = \frac{1}{2}\sum_{e \in V(F)} d_j > |V(F)|.$$

Hence, each term in (2.13) is $o(1)$. Since $|\mathcal{F}| \lesssim_{K,c} 1$, the result in Proposition 2.1 now follows. □

3. Proof of Theorem 1.1

ec:jointTHGnpf⟩ Recall the definition of $T(H, G_n)$ from (1.2), which is the number of monochromatic copies of H in G_n:

$$T(H, G_n) = \frac{1}{|Aut(H)|} \sum_{\boldsymbol{s} \in [n]_{|V(H)|}} \prod_{(a,b) \in E(H)} a_{s_a s_b}(G_n) \mathbf{1}\{X_{=\boldsymbol{s}}\}. \tag{3.1}$$?eq:recall_T_H_

We begin by rewriting $T(H, G_n)$ in terms of $\mathcal{T}(\cdot, \tilde{\boldsymbol{X}})$. To this end, for $J = \{i_1 < \cdots < i_{|J|}\} \subseteq [|V(H)|]$ and $\boldsymbol{s} \in [n]_{|V(H)|}$, denote $\boldsymbol{s}_J = (s_{i_1}, \cdots, s_{i_{|J|}})$.

mial_expansion⟩ **Lemma 3.1.** *For $J = \{i_1 < \cdots < i_{|J|}\} \subseteq [V(H)]$, define $f_{H,J}^{G_n} : [n]_{|J|} \to [0,1]$ as follows:*

$$f_{H,J}^{G_n}(\boldsymbol{t}) := \frac{1}{n^{|V(H)| - |J|}} \sum_{\substack{\boldsymbol{s} \in [n]_{|V(H)|} \\ \boldsymbol{s}_J = \boldsymbol{t}}} \prod_{(a,b) \in E(H)} a_{\boldsymbol{s}_a \boldsymbol{s}_b}(G_n). \tag{3.2}$$ eq:fH

Then

$$T(H,G_n) - \mathbb{E}(T(H,G_n)) = \frac{1}{|Aut(H)|} \sum_{\substack{J \subseteq V(H) \\ |J| \geq 2}} \mathcal{T}\left(f_{H,J}^{G_n}; \tilde{\boldsymbol{X}}\right) \frac{n^{|V(H)| - \frac{|J|}{2}}}{c^{|V(H)| - \frac{|J|}{2} - \frac{1}{2}}}. \tag{3.3}$$

eq:expanded_T_F

Proof. Notice that

$$\begin{aligned}
\mathbf{1}\{X_{=\boldsymbol{s}}\} &= \sum_{a=1}^{c} \prod_{j=1}^{|V(H)|} \mathbf{1}\{X_{s_j} = a\} \\
&= \sum_{a=1}^{c} \prod_{j=1}^{|V(H)|} \left(\mathbf{1}\{X_{s_j} = a\} - \frac{1}{c} + \frac{1}{c} \right) \\
&= \sum_{a=1}^{c} \sum_{J \subseteq V(H)} \prod_{j \in J} \left(\mathbf{1}\{X_{s_j} = a\} - \frac{1}{c} \right) \frac{1}{c^{|V(H)|-|J|}} \\
&= \sum_{a=1}^{c} \sum_{J \subseteq V(H)} \prod_{j \in J} \tilde{X}_{s_j,a} \frac{1}{c^{|V(H)| - \frac{|J|}{2}}}. \qquad (3.4)
\end{aligned}$$

eq:indicator_x_

Then recalling (2.3) we have,

$$\begin{aligned}
\mathcal{T}(f_{H,J}^{G_n}; \tilde{\boldsymbol{X}}) &= \frac{1}{n^{|V(H)| - \frac{|J|}{2}} \sqrt{c}} \sum_{a=1}^{c} \sum_{\boldsymbol{t} \in [n]_{|J|}} \sum_{\substack{\boldsymbol{s} \in [n]_{|V(H)|} \\ \boldsymbol{s}_J = \boldsymbol{t}}} \prod_{(a,b) \in E(H)} a_{\boldsymbol{s}_a \boldsymbol{s}_b}(G_n) \prod_{j \in J} \tilde{X}_{t_j,a} \\
&= \frac{1}{n^{|V(H)| - \frac{|J|}{2}} \sqrt{c}} \sum_{a=1}^{c} \sum_{\boldsymbol{s} \in [n]_{|V(H)|}} \prod_{(a,b) \in E(H)} a_{\boldsymbol{s}_a \boldsymbol{s}_b}(G_n) \prod_{j \in J} \tilde{X}_{s_j,a}. \qquad (3.5)
\end{aligned}$$

eq:THfGn

Hence,

$$\begin{aligned}
&\frac{1}{|Aut(H)|} \sum_{J \subseteq V(H)} \mathcal{T}\left(f_{H,J}^{G_n}; \tilde{\boldsymbol{X}}\right) \frac{n^{|V(H)| - \frac{|J|}{2}}}{c^{|V(H)| - \frac{|J|}{2} - \frac{1}{2}}} \\
&= \frac{1}{|Aut(H)|} \sum_{J \subseteq V(H)} \sum_{a=1}^{c} \sum_{\boldsymbol{s} \in [n]_{|V(H)|}} \prod_{(a,b) \in E(H)} a_{\boldsymbol{s}_a \boldsymbol{s}_b}(G_n) \prod_{j \in J} \tilde{X}_{s_j,a} \frac{1}{c^{|V(H)| - \frac{|J|}{2}}} \\
&= \frac{1}{|Aut(H)|} \sum_{\boldsymbol{s} \in [n]_{|V(H)|}} \prod_{(a,b) \in E(H)} a_{\boldsymbol{s}_a \boldsymbol{s}_b}(G_n) \mathbf{1}\{X_{=\boldsymbol{s}}\}, \qquad (3.6)
\end{aligned}$$

eq:THfGnJsum

where the last step uses (3.4).

Now, observe that, if $J = \{j_0\}$, for some $j_0 \in V(H)$,

$$\sum_{a=1}^{c} \prod_{j \in J} \left(\mathbf{1}\{X_{s_j} = a\} - \frac{1}{c} \right) = \sum_{a=1}^{c} \left(\mathbf{1}\{X_{s_{j_0}} = a\} - \frac{1}{c} \right) = 0.$$

This means (recall (3.5)) that

$$\sum_{\substack{J \subseteq V(H) \\ |J| = 1}} \mathcal{T}\left(f_{H,J}^{G_n}; \tilde{\boldsymbol{X}}\right) = 0. \tag{3.7}$$

eq:THfGnJone

Also, notice that when $J = \varnothing$,

$$\frac{1}{|Aut(H)|}\mathcal{T}\left(f^{G_n}_{H,\varnothing}; \tilde{\boldsymbol{X}}\right)\frac{n^{|V(H)|}}{c^{|V(H)|-\frac{1}{2}}} = \mathbb{E}(T(H, G_n)). \tag{3.8}$$

Combining (3.6), (3.7), and (3.8) and rearranging the sums the result in Lemma 3.1 follows. □

Now, let $\{U_r\}_{r\in V(H)}$ be a collection of i.i.d uniform random variables on $[0, 1]$. Define the counterpart of W_H for finite n as follows:

$$\tilde{W}^{G_n}_H(x, y) := \frac{1}{2}\sum_{1\le u\ne v\le |V(H)|}\mathbb{E}\left(\boldsymbol{1}\{A_n\}\prod_{(i,j)\in E(H)} W^{G_n}(U_i, U_j)\,\middle|\, Z_u = x, Z_v = y\right) \tag{3.9}$$

where $A_n = \{(\lceil nU_r\rceil)_{r\in V(H)} \in V(G_n)_{|V(H)|}\}$ is the event that the variables $\{U_r\}_{r\in V(H)}$ fall in distinct intervals when $[0, 1]$ is partitioned in a grid of size $1/n$. Then we define

$$Q_2(H, G_n) := \frac{1}{n\sqrt{c}}\sum_{a=1}^{c}\sum_{1\le u\ne v\le |V(G_n)|}\tilde{W}^{G_n}_H\left(\frac{u}{n}, \frac{v}{n}\right)\tilde{Z}_{u,a}\tilde{Z}_{v,a}, \tag{3.10}$$

where $\{\tilde{Z}_{v,a}\}_{v\in V(G_n), a\in[c]}$ is defined in (2.2). The following lemma shows that $\Gamma(H, G_n)$ and $Q_2(H, G_n)$ have the same limiting distribution.

Lemma 3.2. *Let $\Gamma(H, G_n)$ and $Q_2(H, G_n)$ be as defined in (1.10) and (3.10), respectively. Then*

$$Q_2(H, G_n) = \mathcal{T}\left(\sum_{\substack{J\subseteq V(H)\\|J|=2}} f^{G_n}_{H,J}; \tilde{\boldsymbol{Z}}\right). \tag{3.11}$$

Also, as $n\to\infty$,

$$\Gamma(H, G_n) = \mathcal{T}\left(\sum_{\substack{J\subseteq V(H)\\|J|=2}} f^{G_n}_{H,J}; \tilde{\boldsymbol{X}}\right) + o_{L_2}(1). \tag{3.12}$$

Proof. Recalling (3.2) and (3.9) gives, for $1 \le u \ne v \le n$,

$$\sum_{\substack{J\subseteq V(H)\\|J|=2}} f^{G_n}_{H,J}(u, v) = \frac{1}{n^{|V(H)|-2}}\sum_{\substack{\boldsymbol{s}\in[n]_{|V(H)|}\\ \boldsymbol{s}_J=\{u,v\}}}\prod_{(a,b)\in E(H)} a_{\boldsymbol{s}_a\boldsymbol{s}_b}(G_n) = \tilde{W}^{G_n}_H(x, y),$$

for $x \in [\frac{u}{n}, \frac{u+1}{n}]$ and $y \in [\frac{v}{n}, \frac{v+1}{n}]$. Therefore,

$$\mathcal{T}\left(\sum_{\substack{J\subseteq V(H)\\|J|=2}} f^{G_n}_{H,J}; \tilde{\boldsymbol{Z}}\right) = \frac{1}{n\sqrt{c}}\sum_{a=1}^{c}\sum_{1\le u\ne v\le |V(G_n)|}\tilde{W}^{G_n}_H\left(\frac{u}{n}, \frac{v}{n}\right)\tilde{Z}_{u,a}\tilde{Z}_{v,a} = Q_2(H, G_n).$$

This proves (3.11).

Next, recalling (1.10) and from (3.3) we have,

$$\Gamma(H, G_n) = \frac{c^{|V(H)|-\frac{3}{2}}}{n^{|V(H)|-1}}\{T(H, G_n) - \mathbb{E}(T(H, G_n))\}$$

$$= \frac{1}{|Aut(H)|} \sum_{\substack{J \subseteq V(H) \\ |J| \geq 2}} \mathcal{T}\left(f_{H,J}^{G_n}; \tilde{\boldsymbol{X}}\right) \frac{c^{\frac{|J|}{2}-\frac{1}{2}}}{n^{\frac{|J|}{2}-1}}. \tag{3.13}$$ eq:GammaHGnJsum

Note that, for any $J \subseteq V(H)$, with $|J| \geq 3$, $\mathbb{E}\mathcal{T}(f_{H,J}^{G_n}; \tilde{\boldsymbol{X}}) = 0$ and, from Lemma 2.1,

$$\frac{1}{n^{|J|-2}} \mathrm{Var}\mathcal{T}(f_{H,J}^{G_n}; \tilde{\boldsymbol{X}}) = \frac{1}{n^{|J|-2}} \|\tilde{f}_{H,J}^{G_n}\|_2^2 = o(1),$$

since

$$\|\tilde{f}_{H,J}^{G_n}\|_2^2 = \frac{1}{n^{2|V(H)|-|J|}} \sum_{\boldsymbol{t} \in [V(G_n)]_{|J|}} \left(\frac{1}{|J|!} \sum_{\sigma \in \mathcal{S}_{|J|}} \sum_{\substack{\boldsymbol{s} \in [V(G_n)]_{|V(H)|} \\ \boldsymbol{s}_J = \sigma(\boldsymbol{t})}} \prod_{(a,b) \in E(H)} a_{\boldsymbol{s}_a \boldsymbol{s}_b}(G_n) \right)^2 \leq 1,$$

where $\sigma(\boldsymbol{t}) = (t_{\sigma(1)}, \ldots, t_{\sigma(|J|)})$. Hence, $\mathcal{T}(f_{H,J}^{G_n}) = o_{L_2}(1)$, for all $J \subseteq$ such that $|J| \geq 3$. This implies, from (3.13),

$$\Gamma(H, G_n) = \frac{c^{|V(H)|-\frac{3}{2}}}{n^{|V(H)|-1}} \sum_{\substack{J \subseteq V(H) \\ |J| = 2}} \mathcal{T}\left(f_{H,J}^{G_n}; \tilde{\boldsymbol{X}}\right) \frac{n^{|V(H)|-\frac{|J|}{2}}}{c^{|V(H)|-\frac{|J|}{2}-\frac{1}{2}}} + o_{L_2}(1)$$

$$= \mathcal{T}\left(\sum_{\substack{J \subseteq V(H) \\ |J| = 2}} f_{H,J}^{G_n}; \tilde{\boldsymbol{X}} \right) + o_{L_2}(1). \tag{3.14}$$ eq:Gamma_eq_Gam

This proves (3.14) and completes the proof of Lemma 3.2. □

The following lemma represents $Q_2(H, G_n)$ as a sum of independent bivariate stochastic integrals.

o_independent⟩? **Lemma 3.3.** *Let $\{B^{(1)}(t), B^{(2)}(t), \ldots, B^{(c-1)}(t)\}_{t \in [0,1]}$ be a collection of independent Brownian motions on $[0, 1]$ and $Q_2(H, G_n)$ be as defined in (3.10). Then*

$$Q_2(H, G_n) \stackrel{D}{=} \frac{1}{\sqrt{c}} \sum_{a=1}^{c-1} \int_{[0,1]^2} \tilde{W}_H^{G_n}(x, y) \mathrm{d}B_x^{(a)} \mathrm{d}B_y^{(a)}.$$

Proof. Let $\{Z_{v,a}\}_{v \in [n], a \in [c]}$ be a collection of i.i.d $N(0, 1)$ random variables. For $1 \leq v \leq n$, let $\boldsymbol{Z}_v = (Z_{v,1}, Z_{v,2}, \ldots, Z_{v,c})^\top$. Then, for $1 \leq v \leq n$, $\boldsymbol{M}\boldsymbol{Z}_v = \tilde{\boldsymbol{Z}}_v$, where $\tilde{\boldsymbol{Z}}_v = (\tilde{Z}_{v,1}, \tilde{Z}_{v,2}, \ldots, \tilde{Z}_{v,c})^\top$, where $\boldsymbol{M} = \boldsymbol{I} - \frac{1}{c}\boldsymbol{1}^\top \boldsymbol{1}$, where $\boldsymbol{1}$ is the vector of ones of length c. Note that $\boldsymbol{M}^\top \boldsymbol{M} = \boldsymbol{M}$ and $\boldsymbol{M}$ has $c - 1$ non-zero eigenvalues all of which are 1. Hence, we can write,

$$Q_2(H, G_n) = \frac{1}{n\sqrt{c}} \sum_{a=1}^{c} \sum_{1 \leq u \neq v \leq |V(G_n)|} \tilde{W}_H^{G_n}\left(\frac{u}{n}, \frac{v}{n}\right) \tilde{Z}_{u,a} \tilde{Z}_{v,a}$$

$$= \frac{1}{n\sqrt{c}} \sum_{1 \leq u \neq v \leq |V(G_n)|} \tilde{W}_H^{G_n}\left(\frac{u}{n}, \frac{v}{n}\right) \boldsymbol{Z}_u^\top \boldsymbol{M}^\top \boldsymbol{M} \boldsymbol{Z}_v$$

$$= \frac{1}{n\sqrt{c}} \sum_{1 \leq u \neq v \leq |V(G_n)|} \tilde{W}_H^{G_n}\left(\frac{u}{n}, \frac{v}{n}\right) \boldsymbol{Z}_u^\top \boldsymbol{M} \boldsymbol{Z}_v$$

$$\stackrel{D}{=} \frac{1}{n\sqrt{c}} \sum_{1 \le u \neq v \le |V(G_n)|} \tilde{W}_H^{G_n}\left(\frac{u}{n}, \frac{v}{n}\right) \sum_{a=1}^{c-1} Y_{u,a} Y_{v,a}$$

(where $\{Y_{v,a}\}_{v \in [n], a \in [c]}$ are i.i.d. $N(0,1)$)

$$\stackrel{D}{=} \frac{1}{\sqrt{c}} \sum_{a=1}^{c-1} \int_{[0,1]^2} \tilde{W}_H^{G_n}(x,y) \mathrm{d}B_x^{(a)} \mathrm{d}B_y^{(a)},$$

where the last equality holds in distribution because $\tilde{W}_H^{G_n}(x,y)$ is a step function which is zero on the diagonal. □

With the above preparations, we proceed with the proof of Theorem 1.1. For this, suppose $\mathcal{H} = \{H_1, H_2, \ldots, H_d\}$ is a finite collection of graphs and $\boldsymbol{G}_n = (G_n^{(1)}, G_n^{(2)}, \ldots, G_n^{(d)})$ be a sequence of d-multiplexes converging in joint cut-norm to $\boldsymbol{W} = (W_1, W_2, \ldots, W_d)$. By Lemma 3.2,

$$\boldsymbol{\Gamma}(\mathcal{H}, \boldsymbol{G}_n) = \boldsymbol{Q}_2(\mathcal{H}, \boldsymbol{G}_n) + o_{L_2}(1),$$

where $\boldsymbol{\Gamma}(\mathcal{H}, \boldsymbol{G}_n)$ is defined in (1.9) and

$$\boldsymbol{Q}_2(\mathcal{H}, \boldsymbol{G}_n) = (Q_2(H_1, G_n^{(1)}), Q_2(H_1, \ldots, Q_2(H_d, G_n^{(d)}))^\top.$$

Therefore, to prove Theorem 1.1 it suffices to derive the limiting distribution of $\boldsymbol{Q}_2(\mathcal{H}, \boldsymbol{G}_n)$. To this end, invoking the Cramér-Wold device, it suffices to derive the limiting distribution of $\boldsymbol{\alpha}^\top \boldsymbol{Q}_2(\mathcal{H}, \boldsymbol{G}_n)$, for any vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{R}^d$. Note that

$$\boldsymbol{\alpha}^\top \boldsymbol{Q}_2(\mathcal{H}, \boldsymbol{G}_n) = \sum_{i=1}^{d} \alpha_i Q_2(H_i, G_n^{(i)}) = \frac{1}{\sqrt{c}} \sum_{a=1}^{c-1} I_2^{(a)} \left(\sum_{i=1}^{d} \alpha_i \tilde{W}_{H_i}^{G_n^{(i)}} \right), \tag{3.15}$$ eq:sum_of_stoch

where $I_2^{(a)}(F) := \int_{[0,1]^2} F(x,y) \mathrm{d}B_x^{(a)} \mathrm{d}B_y^{(a)}$, for any bounded symmetric kernel $F \in \mathscr{W}_1$. Observe that for any $1 \le i \le d$,

$$\|\tilde{W}_{H_i}^{G_n^{(i)}} - W_{H_i}^{G_n^{(i)}}\|_2^2 \le |V(H_i)|^2 \mathbb{P}(A_n^c).$$

Since $(\lceil n Z_u \rceil)_{u \in V(H)}$ is a uniform random vector over $V(G_n)^{V(H)}$,

$$\mathbb{P}(A_n) = \frac{n(n-1)\cdots(n-|V(H)|)}{n^{|V(H)|}} \to 1.$$

Hence, (3.15) has the same limiting distribution as

$$\frac{1}{\sqrt{c}} \sum_{a=1}^{c-1} I_2^{(a)} \left(\sum_{i=1}^{d} \alpha_i W_{H_i}^{G_n^{(i)}} \right). \tag{3.16}$$ eq:sum_of_stoch

Now, let

$$W_{\boldsymbol{\alpha},n} := \sum_{i=1}^{d} \alpha_i W_{H_i}^{G_n^{(i)}} \text{ and } W_{\boldsymbol{\alpha}} := \sum_{i=1}^{d} \alpha_i W_{H_i}^{(i)}. \tag{3.17}$$ eq:WGnalpha

Since $\boldsymbol{G}_n$ converges to $\boldsymbol{W}$ in joint cut-metric, there is a sequence of invertible measure preserving maps $\phi_n : [0,1] \to [0,1]$ such that

$$\sum_{i=1}^{d} \|(W^{G_n^{(i)}})^{\phi_n} - W^{(i)}\|_\square \to 0.$$

Define the map $\eta_{H_i} : W \to W_{H_i}$, for $1 \leq i \leq d$. Notice that $\eta_{H_i}((W^{G_n^{(i)}})^{\phi_n}) = \eta_{H_i}(W^{G_n^{(i)}})^{\phi_n}$. Then by Lemma A.1, as $n \to \infty$,

$$\|W^{\phi_n}_{\boldsymbol{\alpha},n} - W_{\boldsymbol{\alpha}}\|_\square \to 0. \tag{3.18}$$

eq:Walphaconver

Using (3.17) we can rewrite (3.16) as

$$\begin{aligned}
\frac{1}{\sqrt{c}} \sum_{a=1}^{c-1} I_2^{(a)} \left(\sum_{i=1}^{d} \alpha_i W^{G_n^{(i)}}_{H_i} \right) &= \frac{1}{\sqrt{c}} \sum_{a=1}^{c-1} I_2^{(a)} \left(W_{\boldsymbol{\alpha},n} \right) \\
&\stackrel{D}{=} \frac{1}{\sqrt{c}} \sum_{a=1}^{c-1} I_2^{(a)} \left(W^{\phi_n}_{\boldsymbol{\alpha},n} \right) \\
&= \frac{1}{\sqrt{c}} \sum_{a=1}^{c-1} I_2^{(a)} (W^{\phi_n}_{\boldsymbol{\alpha},n} - W_{\boldsymbol{\alpha}}) + \frac{1}{\sqrt{c}} \sum_{a=1}^{c-1} I_2^{(a)} (W_{\boldsymbol{\alpha}}). \qquad (3.19)
\end{aligned}$$

eq:QHGnWlinear

By Lemma B.1 the two terms in the RHS above are asymptotically independent, hence, it suffices to derive the limiting distribution of the first term.

⟨lm:I2clt⟩ **Lemma 3.4.** *As $n \to \infty$,*

$$\frac{1}{\sqrt{c}} \sum_{a=1}^{c-1} I_2^{(a)} (W^{\phi_n}_{\boldsymbol{\alpha},n} - W_{\boldsymbol{\alpha}}) \stackrel{D}{\to} \sqrt{2\left(1 - \frac{1}{c}\right)} N(0, \boldsymbol{\alpha}^\top \Sigma \boldsymbol{\alpha}), \tag{3.20}$$

eq:l2clt

where Σ is defined in Theorem 1.1.

Proof. By the spectral theorem for bounded symmetric kernels (see [38, Section 7.5]),

$$\Delta_{\alpha,n}(x, y) := W^{\phi_n}_{\boldsymbol{\alpha},n}(x, y) - W_{\boldsymbol{\alpha}}(x, y) = \sum_{s=1}^{\infty} \lambda_s^{(n)} \phi_s^{(n)}(x) \phi_s^{(n)}(y),$$

where $\{\lambda_s^{(n)}\}_{s \geq 1}$ are the eigenvalues and $\{\phi_s^{(n)}\}_{s \geq 1}$ are a set of orthonormal eigenvectors for the operator

$$T_{\Delta_{\boldsymbol{\alpha},n}} f(x) = \int_0^1 \Delta_{\boldsymbol{\alpha},n}(x, y) f(y) \mathrm{d}y.$$

Define, for $s \geq 1$,

$$\xi_s = \sum_{a=1}^{c-1} \left(\int_0^1 \phi_s^{(n)}(x) \mathrm{d}B_x^{(a)} \right)^2 - (c - 1).$$

By orthonormality, the variables $\{\xi_s\}_{s \geq 1}$ have independent $\chi^2_{c-1} - (c - 1)$ distributions. Hence,

$$\frac{1}{\sqrt{c}} \sum_{a=1}^{c-1} I_2^{(a)} (W^{\phi_n}_{\boldsymbol{\alpha},n} - W_{\boldsymbol{\alpha}}) \stackrel{D}{=} \frac{1}{\sqrt{c}} \sum_{s=1}^{\infty} \lambda_s^{(n)} \xi_s.$$

The result in (3.20) then follows from Lemma C.1, if we show the following:

$$\lim_{n \to \infty} \max_{s \geq 1} |\lambda_s^{(n)}| = 0 \text{ and } \lim_{n \to \infty} \sum_{s \geq 1} |\lambda_s^{(n)}|_2^2 = \boldsymbol{\alpha}^\top \Sigma \boldsymbol{\alpha}. \tag{3.21}$$

eq:lambdamaxsur

To show the first condition in (3.21), we will use the identity $\sum_{s\geq 1}(\lambda_s^{(n)})^4 = t(C_4, \Delta_{n,\boldsymbol{\alpha}})$ (see [38, Section 7.5]), where C_4 denotes the cycle of size 4. This implies,

$$\max_{s\geq 1}|\lambda_s^{(n)}|^4 \leq \sum_{s=1}^{\infty}(\lambda_s^{(n)})^4 = t(C_4, \Delta_{n,\boldsymbol{\alpha}}) \leq \|\Delta_{n,\boldsymbol{\alpha}}\|_\square \to 0,$$

by (3.18).

We now prove the second condition in (3.21). For this, observe that

$$\begin{aligned}\lim_{n\to\infty}\|\Delta_{n,\boldsymbol{\alpha}}\|_2^2 &= \lim_{n\to\infty}\|W_{\boldsymbol{\alpha},n}^{\phi_n} - W_{\boldsymbol{\alpha}}\|_2^2 \\ &= \lim_{n\to\infty}\|W_{\boldsymbol{\alpha},n}\|_2^2 - 2\lim_{n\to\infty}\langle W_{\boldsymbol{\alpha},n}, W_{\boldsymbol{\alpha}}\rangle + \|W_{\boldsymbol{\alpha}}\|_2^2 \\ &= \lim_{n\to\infty}\|W_{\boldsymbol{\alpha},n}\|_2^2 - \|W_{\boldsymbol{\alpha}}\|_2^2,\end{aligned}$$

using the fact that $\lim_{n\to\infty}\langle W_{\boldsymbol{\alpha},n}, W_{\boldsymbol{\alpha}}\rangle = \|W_{\boldsymbol{\alpha}}\|_2^2$, which follows from (3.18) and by invoking [38, Lemma 8.22]. Also,

$$\begin{aligned}\lim_{n\to\infty}\|W_{\boldsymbol{\alpha},n}\|_2^2 &= \lim_{n\to\infty}\left\|\sum_{i=1}^d \alpha_i W_{H_i}^{G_n^{(i)}}\right\|_2^2 \\ &= \lim_{n\to\infty}\left\{\sum_{i=1}^d \alpha_i^2\|W_{H_i}^{G_n^{(i)}}\|_2^2 + \sum_{1\leq i\neq j\leq d}\alpha_i\alpha_j\langle W_{H_i}^{G_n^{(i)}}, W_{H^{(j)}}^{G_n^{(j)}}\rangle\right\} \\ &= \sum_{i=1}^d \alpha_i^2\tau(H_i, W_i) + \sum_{1\leq i\neq j\leq d}\alpha_i\alpha_j\rho_{ij},\end{aligned}$$

where $\tau(H_i, W_i)$ is as defined in Lemma 3.5 and ρ_{ij} is as in (1.12). Combining (**??**), (**??**), and observing that

$$\|W_{\boldsymbol{\alpha}}\|_2^2 = \sum_{i=1}^d \alpha_i^2\|W_{H_i}^{(i)}\|_2^2 + \sum_{1\leq i\neq j\leq d}\alpha_i\alpha_j\langle W_{H_i}^{(i)}, W_{H^{(j)}}^{(j)}\rangle,$$

the second condition in (3.21) follows. This completes the proof of Lemma 3.4. □

The proof of Lemma 3.4 uses the following result about the limiting value of $\|W_H^{G_n}\|_2^2$.

m:WGnHvariance⟩ **Lemma 3.5.** *For any fixed graph $H = (V(H), E(H))$ and a sequence of graphs $\{G_n\}_{n\geq 1}$ converging to a graphon W,*

$$\lim_{n\to\infty}\|W_H^{G_n}\|_2^2 = \frac{1}{4}\sum_{\substack{1\leq u\neq v\leq |V(H)| \\ 1\leq u'\neq v'\leq |V(H)|}} t\left(H \bigotimes_{(u,v),(u',v')} H, W\right) := \tau(H, W).$$

Moreover, gor any two fixed graphs $H_1 = (V(H_1), E(H_1))$ and $H_2 = (V(H_2), E(H_2))$ and a sequence of graphs $\{G_n\}_{n\geq 1}$ converging to a graphon W,

$$\lim_{n\to\infty}\langle W_{H_1}^{G_n}, W_{H_2}^{G_n}\rangle = \frac{1}{4}\sum_{\substack{1\leq u\neq v\leq |V(H_1)| \\ 1\leq u'\neq v'\leq |V(H_2)|}} t\left(H_1 \bigotimes_{(u,v),(u',v')} H_2, W\right). \quad (3.22)$$

lm:WGnHcovarian

Proof. Let H' be an isomorphic copy of H. Then

$$\|W_H^{G_n}\|_2^2 = \int_{[0,1]^2} \left[\frac{1}{2} \sum_{1 \le a \ne b \le |V(H)|} \mathbb{E} \left(\prod_{(i,j) \in E(H)} W^{G_n}(U_i, U_j) \,\middle|\, U_a = x, U_b = y \right) \right]^2 \mathrm{d}x\mathrm{d}y$$

$$= \int_{[0,1]^2} \frac{1}{4} \sum_{\substack{(a,b) \in V(H)_2 \\ (a',b') \in V(H')_2}} \mathbb{E} \left(\prod_{(i,j) \in E(H) \cup E(H')} W^{G_n}(U_i, U_j) \,\middle|\, \begin{pmatrix} U_a \\ U_b \end{pmatrix} = \begin{pmatrix} U_{a'} \\ U_{b'} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \right) \mathrm{d}x\mathrm{d}y$$

$$= \frac{1}{4} \sum_{\substack{1 \le a \ne b \le |V(H)| \\ 1 \le a' \ne b' \le |V(H)|}} t\left(H \bigotimes_{(a,b),(a',b')} H, W^{G_n} \right) \to \tau(H, W).$$

The result in (3.22) can be proved similarly. □

To complete the proof of Theorem 1.1, recalling the representation in (3.19). Then applying Lemma 3.4 and Lemma B.1 shows that

$$\frac{1}{\sqrt{c}} \sum_{a=1}^{c-1} I_2^{(a)} \left(\sum_{i=1}^{d} \alpha_i W_{H_i}^{G_n^{(i)}} \right) \xrightarrow{D} \sqrt{2\left(1 - \frac{1}{c}\right)} \boldsymbol{\alpha}^\top \boldsymbol{Z} + \frac{1}{\sqrt{c}} \sum_{a=1}^{c-1} I_2^{(a)}(W_{\boldsymbol{\alpha}}),$$

where $\boldsymbol{Z}$ is as defined Theorem 1.1 and the 2 terms in the RHS are independent. Finally, recall (from (3.15) and (3.16)) that the LHS above has the same limiting distribution as $\boldsymbol{\alpha}^\top \boldsymbol{Q}_2(\mathcal{H}, \boldsymbol{G}_n)$. Hence, by the Cramér-Wold device the result in Theorem 1.1 follows. □

4. Examples

⟨sec:examples⟩ In this section we illustrate our results in various examples. A few of the examples will involve random multiplexes. To this end, it is worth noting that Theorems 1.1 can be easily extended to random multiplexes, when the limits in (1.8) and (1.12) hold in probability, under the assumption that the multiplex and its coloring are jointly independent (see for example Lemma in []).

d_erdos_renyi⟩? **Example 4.1.** A natural example of a random multiplex is the *correlated Erdős-Rényi model* $\boldsymbol{G}(n, p, q, \rho)$, which is a 2-multiplex where the edges are dependent across the different layers. This model emerged from the study of network privacy [55] and is the basic underlying model in graph matching problems (see [42, 43] and the references therein). Specifically, $\boldsymbol{G}(n, p, q, \rho) = (G_n^{(1)}, G_n^{(2)})$ is a 2-multiplex with common vertex set $[n]$, where independently for every $1 \le i < j \le n$, we have

$$\mathbb{P}((i,j) \in E(G_n^{(1)})) = p, \quad \mathbb{P}((i,j) \in E(G_n^{(2)})) = q$$

and

$$\mathbb{P}((i,j) \in E(G_n^{(1)}) \cap E(G_n^{(2)})) = \rho + pq := p_{1,2},$$

for $p, q \in (0,1)$ and $\rho \in [0, \min\{p,q\} - pq)$. In this case, $(G_n^{(1)}, G_n^{(2)})$ converges jointly to (W_1, W_2), where $W_1 \equiv p$ and $W_2 \equiv q$. Therefore, for any 2 graphs H_1 and H_2,

$$((W_1)_{H_1}, (W_2)_{H_2}) = \left(\binom{|V(H_1)|}{2} p^{|E(H_1)|}, \binom{|V(H_2)|}{2} q^{|E(H_2)|} \right).$$

Moreover,

$$\lim_{n \to \infty} \langle W_{H_1}^{G_n^{(1)}}, W_{H_2}^{G_n^{(2)}} \rangle = \rho_{12},$$

in probability. Then by Theorem 1.1, under the assumption that the coloring is independent of the multiplex,

$$\begin{pmatrix} \Gamma(H_1, G_n^{(1)}) \\ \Gamma(H_2, G_n^{(2)}) \end{pmatrix} \xrightarrow{D} \sqrt{2\left(1-\frac{1}{c}\right)} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} + \frac{1}{\sqrt{c}} \begin{pmatrix} \binom{|V(H_1)|}{2} p^{|E(H_1)|} \\ \binom{|V(H_2)|}{2} q^{|E(H_2)|} \end{pmatrix} \xi,$$

with $\xi \sim \chi^2_{c-1} - (c-1)$ which is independent of $(Z_1, Z_2) \sim N_2(\mathbf{0}, \Sigma)$ where

$$\Sigma = \begin{pmatrix} p^{2|E(H_1)|-1}(1-p)|E(H_1)|^2 & \rho \cdot p^{|E(H_1)|-1} q^{|E(H_2)|-1} |E(H_1)||E(H_2)| \\ \rho \cdot p^{|E(H_1)|-1} q^{|E(H_2)|-1} |E(H_1)||E(H_2)| & q^{2|E(H_2)|-1}(1-q)|E(H_2)|^2 \end{pmatrix}.$$

Observe that when $\rho = 0$, that is, the 2 layers are independent, the covariance matrix Σ becomes diagonal.

Next, we construct an example where the Gaussian component of the limit has a degenerate covariance matrix.

_complementar⟩? **Example 4.2.** Suppose $G_n^{(1)} \sim G(n,p)$, where $p \in (0,1)$, and consider the multiplex $\boldsymbol{G}_n = (G_n^{(1)}, G_n^{(2)})$, where $G_n^{(2)}$ is the complement graph of $G_n^{(1)}$. Then, by Theorem 1.1, when $c = 2$,

$$\begin{pmatrix} \Gamma(K_3, G_n^{(1)}) \\ \Gamma(K_3, G_n^{(2)}) \end{pmatrix} \xrightarrow{D} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} + \frac{3}{\sqrt{2}} \begin{pmatrix} p^3 \\ (1-p)^3 \end{pmatrix} \xi,$$

with $\xi \sim \chi^2_1 - 1$ which is independent of $(Z_1, Z_2) \sim N_2(\mathbf{0}, \Sigma)$, where

$$\Sigma = \begin{pmatrix} 9p^5(1-p) & -9p^3(1-p)^3 \\ -9p^3(1-p)^3 & 9(1-p)^5 p \end{pmatrix}.$$

Note that the rank of Σ is 1.

The next example shows that marginal convergence (in the cut-distance) of the graphs in the different layers of a multiplex $\boldsymbol{G}_n$ and the convergence of the pairwise overlaps (1.12) are not enough for the convergence of the joint distribution of $\Gamma(\mathcal{H}, \boldsymbol{G}_n)$. One needs to assume that the graphs in the different layers converge jointly in the cut-distance (as in (1.8)) for the limiting distribution of $\Gamma(\mathcal{H}, \boldsymbol{G}_n)$ to exist.

ence_necessary⟩ **Example 4.3.** Let A_1, A_2, A_3, A_4 be 4-disjoint sets of size n. Denote by $B_n^{(s,s+1)}$ the complete bipartite graph between the sets A_s and A_{s+1}, for $1 \leq s \leq 3$ (see Figure 2). Define[3]

$$G_n^{(1)} = \bigcup_{s=1}^{3} B_n^{(s,s+1)}, \quad G_n^{(2)} = B_n^{(1,2)}, \text{ and } G_n^{(3)} = B_n^{(2,3)}.$$

(Note that, in other words, G_n is the n-blow-up of a path with 4 vertices [].) Now consider the following 2 sequences of the multiplexes:

$$\boldsymbol{G}_n = (G_n^{(1)}, G_n^{(2)}) \text{ and } \tilde{\boldsymbol{G}}_n = (G_n^{(1)}, G_n^{(3)}).$$

Note that the graph $G_n^{(1)}$ converges to the graphon W_{1234} in Figure 3 (a) and the graphs $G_n^{(2)}$ and $G_n^{(3)}$ converge to the graphon W_{23} in Figure 3 (b). Hence, marginally the graphs in the 2 layers of

[3] For 2 graphs $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$, denote by $G_1 \bigcup G_2 = (V(G_1) \bigcup V(G_2), E(G_1) \bigcup E(G_2))$.

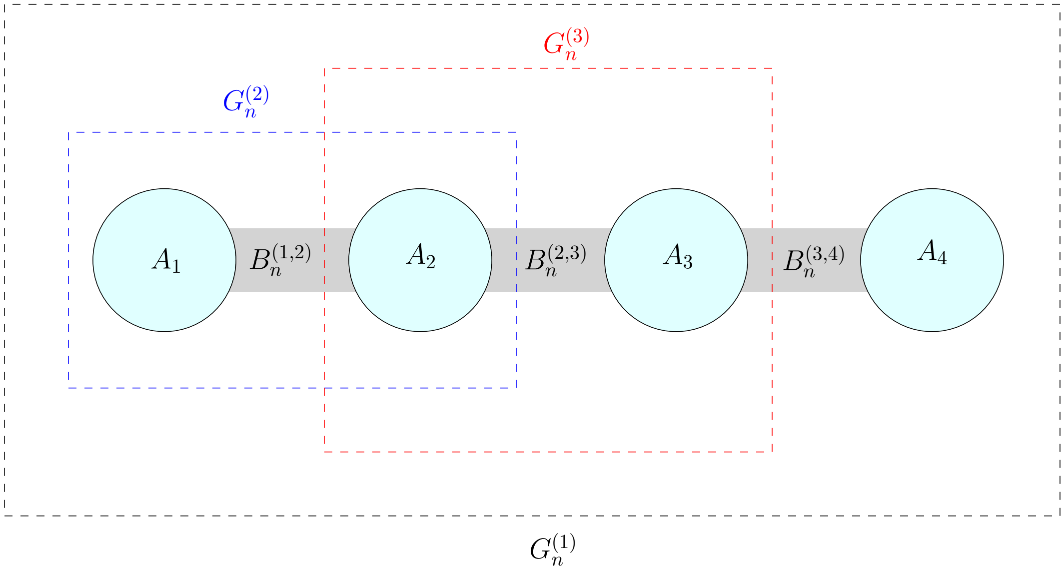

FIGURE 2. The graphs $G_n^{(1)}$, $G_n^{(2)}$, and $G_n^{(3)}$ in Example 4.3.

ig:graphlayers⟩

$\boldsymbol{G}_n$ and and the graphs in the 2 layers of $\tilde{\boldsymbol{G}}_n$ converge to the same limits. Moreover,

$$\lim_{n\to\infty} \frac{2|E(G_n^{(1)}) \cap E(G_n^{(2)})|}{n^2} = \lim_{n\to\infty} \frac{2|E(G_n^{(1)}) \cap E(G_n^{(3)})|}{n^2} = \frac{1}{8},$$

that is, (1.13) converges to the same limit for both $\boldsymbol{G}_n$ and $\tilde{\boldsymbol{G}}_n$.

FIGURE 3. The graphons W_{1234} and W_{23} in Example 4.3.

partiteexample⟩

Hence, the premises of Corollary 1.2 hold for both $\boldsymbol{G}_n$ and $\tilde{\boldsymbol{G}}_n$. In particular, with $c = 2$, we get

$$\begin{pmatrix} \Gamma(K_2, G_n^{(1)}) \\ \Gamma(K_2, G_n^{(2)}) \end{pmatrix} \xrightarrow{D} \begin{pmatrix} \int_{[0,1]^2} W_{1234}(x,y) \mathrm{d}B_x \mathrm{d}B_y \\ \int_{[0,1]^2} W_{23}(x,y) \mathrm{d}B_x \mathrm{d}B_y \end{pmatrix} \overset{D}{=} \frac{1}{4} \begin{pmatrix} Z_1Z_2 + Z_2Z_3 + Z_3Z_4 \\ Z_1Z_2 \end{pmatrix} =: \begin{pmatrix} Y \\ Y' \end{pmatrix} \tag{4.1}$$

eq:Gn12

and

$$\begin{pmatrix}\Gamma(K_2, G_n^{(1)})\\ \Gamma(K_2, G_n^{(3)})\end{pmatrix} \xrightarrow{D} \begin{pmatrix}\int_{[0,1]^2} W_{1234}(x,y)\mathrm{d}B_x\mathrm{d}B_y\\ \int_{[0,1]^2} W_{23}(x,y)\mathrm{d}B_x\mathrm{d}B_y\end{pmatrix} \stackrel{D}{=} \frac{1}{4}\begin{pmatrix}Z_1Z_2 + Z_2Z_3 + Z_3Z_4\\ Z_2Z_3\end{pmatrix} =: \begin{pmatrix}Y\\ Y''\end{pmatrix}, \tag{4.2}$$ eq:Gn13

where Z_1, Z_2, Z_3, Z_4 are i.i.d. $N(0,1)$ and $\{B_t\}_{t\in[0,1]}$ is the standard Brownian Motion on $[0,1]$. Note that the limiting distributions in (4.1) and (4.2). In particular, $\mathbb{E}(Y - Y')^4 = \frac{36}{4^4}$ which is different from $\mathbb{E}(Y - Y'')^4 = \frac{24}{4^4}$. The reason for this is that there is no common permutation of $[n]$ for which $(G_n^{(1)}, G_n^{(2)}, G_n^{(3)})$ jointly converges in the cut-distance.

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Appendix A. Continuity of the 2-Point Conditional Kernel

Recall the definition of the 2-point conditional kernel from (1.11). Note that W_H is symmetric and bounded, but can take values greater 1, hence, $W_H \in \mathscr{W}_1$, where $\mathscr{W}_1$ is the space of bounded, symmetric, measurable functions from $[0,1]^2 \to [0,\infty)$. The topology generated by the cut-distance (1.7) (extended naturally to the space $\mathscr{W}_1$) is the same as the topology generated by the following norm (see [38, Lemma 8.11]):

$$||W_1 - W_2||_{1\to\infty} := \sup_{f,g:[0,1]\to[-1,1]} \left| \int_{[0,1]^2} \left(W_1(x,y) - W_2(x,y)\right) f(x)g(y)\mathrm{d}x\mathrm{d}y \right|,$$

for $W_1, W_2 \in \mathscr{W}_1$. The next lemma shows that the map $W \to W_H$ is Lipschitz in the $\|\cdot\|_{1\to\infty}$ norm and, hence, the cut-distance $\|\cdot\|_\square$. This, in particular, means that if $\{G_n\}_{n\geq 1}$ converges to a graphon W, then $W_H^{G_n}$ converges in cut-norm to W_H.

:W_H_lipschitz⟩ **Lemma A.1.** *The mapping $\eta_H : W \mapsto W_H$ defined on $\mathscr{W}_1$ is Lipschitz under $\|\cdot\|_{\infty\to 1}$ and, hence, under $\|\cdot\|_\square$.*

Proof. Let $\{U_r\}_{r\in V(H)}$ be a collection i.i.d. Unif$[0,1]$ random variables. To show the map η_H is Lipschitz, it suffices to prove

$$\|(W+R)_H - R_H\|_{\infty\to 1} \lesssim_H \|R\|_{\infty\to 1}, \tag{A.1}$$ eq:WRHsquare

for $W, R \in \mathscr{W}_1$. To this end, fix $1 \leq a \neq b \leq |V(H)|$ and note that

$$\begin{aligned}\Delta_{a,b}(x,y) &:= \mathbb{E}\left(\prod_{(i,j)\in E(H)} (W(U_i,U_j) + R(U_i,U_j)) - \prod_{(i,j)\in E(H)} W(U_i,U_j) \middle| U_a = x, U_b = y\right)\\ &= \mathbb{E}\left(\sum_{\substack{\mathcal{A}\subset E(H)\\ |\mathcal{A}|\geq 1}} \prod_{(i,j)\in\mathcal{A}^c} W(U_i,U_j) \prod_{(i,j)\in\mathcal{A}} R(U_i,U_j) \middle| U_a = x, U_b = y\right).\end{aligned}$$

Recalling (1.11), note that,

$$\|(W+R)_H - R_H\|_{\infty\to 1} = \left\| \sum_{1\leq a\neq b\leq |V(H)|} \Delta_{a,b}(x,y) \right\|_{\infty\to 1} \leq \sum_{1\leq a\neq b\leq |V(H)|} \|\Delta_{a,b}(x,y)\|_{\infty\to 1}, \tag{A.2}$$ eq:Deltaxy

by the triangle inequality. Hence, to prove (A.1) it suffices to consider each term in the above sum separately. Towards this,

$$\begin{aligned}&\|\Delta_{a,b}(x,y)\|_{\infty\to 1}\\ &= \sup_{\substack{\|f\|_\infty\leq 1\\ \|g\|_\infty\leq 1}} \left| \mathbb{E}\left(\mathbb{E}\left(\sum_{\substack{\mathcal{A}\subset E(H)\\ |\mathcal{A}|\geq 1}} \prod_{(i,j)\in\mathcal{A}^c} W(U_i,U_j) \prod_{(i,j)\in\mathcal{A}} R(U_i,U_j) \middle| U_a, U_b \right) f(U_a) g(U_b) \right) \right|\\ &= \sup_{\substack{\|f\|_\infty\leq 1\\ \|g\|_\infty\leq 1}} \left| \mathbb{E}\left(\sum_{\substack{\mathcal{A}\subset E(H)\\ |\mathcal{A}|\geq 1}} f(U_a) g(U_b) \prod_{(i,j)\in\mathcal{A}^c} W(U_i,U_j) \prod_{(i,j)\in\mathcal{A}} R(U_i,U_j) \right) \right|\\ &= \sup_{\substack{\|f\|_\infty\leq 1\\ \|g\|_\infty\leq 1}} \left| \sum_{\substack{\mathcal{A}\subset E(H)\\ |\mathcal{A}|\geq 1}} \mathbb{E}\left(f(U_a) g(U_b) \prod_{(i,j)\in\mathcal{A}^c} W(U_i,U_j) \prod_{(i,j)\in\mathcal{A}} R(U_i,U_j) \right) \right|\\ &\leq \sum_{\substack{\mathcal{A}\subset E(H)\\ |\mathcal{A}|\geq 1}} \sup_{\substack{\|f\|_\infty\leq 1\\ \|g\|_\infty\leq 1}} \left| \mathbb{E}\left(f(U_a) g(U_b) \prod_{(i,j)\in\mathcal{A}^c} W(U_i,U_j) \prod_{(i,j)\in\mathcal{A}} R(U_i,U_j) \right) \right|, \end{aligned} \tag{A.3}$$ eq:Deltaxyfg

by the triangle inequality. Now, by a telescoping argument similar to the proof of the counting lemma (see [16, Theorem 3.7] and [38, Lemma 10.24]) it can be shown that

$$\sup_{\substack{\|f\|_\infty \leq 1 \\ \|g\|_\infty \leq 1}} \left| \mathbb{E}\left(f(U_a)g(U_b) \prod_{(i,j)\in\mathcal{A}^c} W(U_i,U_j) \prod_{(i,j)\in\mathcal{A}} R(U_i,U_j) \right) \right| \lesssim_H \|R\|_{\infty\to 1}. \tag{A.4}$$ eq:WAfg

Combining (A.2), (A.3), and (A.4), the result in (A.1) follows. □

Appendix B. Independence of Bivariate Stochastic Integrals

Fix $d \geq 1$. Given a bounded function $f : [0,1]^d \to \mathbb{R}$, denote by $I_d(f)$ the d-dimensional Wiener-Itô stochastic integral of f with respect to a Brownian motion on $[0,1]$. Conditions under which 2 multiple stochastic integrals are independent are well-known. Üstünel and Zakai [62] provided a useful necessary and sufficient for the independence of multiple stochastic integrals and Rosiński and Samorodnitsky [57] showed that multiple stochastic integrals are independent if and only if their squares are uncorrelated. An asymptotic version of these results was established by Nourdin and Rosiński [51]. Using this asymptotic result we prove the following lemma:

independent_L2⟩ **Lemma B.1.** *Fix $W \in \mathscr{W}_1$ and consider a sequence of bounded kernels $\{R_n\}_{n\geq 1}$, with $R_n \in \mathscr{W}_1$, such that $\|U_n\|_\square \to 0$. Then $\boldsymbol{Q}_n := (I_2(R_n), I_2(W))$ are asymptotically independent along any subsequence for which $\boldsymbol{Q}_n$ has a limit in distribution.*

Proof. Note that by definition $\mathbb{E} I_2(R_n) = 0$ and $\operatorname{Var} I_2(R_n) = O(1)$, since R_n is bounded. This means $\boldsymbol{Q}_n$ converges in distribution along a subsequence. Hence, by [51, Theorem 3.1] to show that the asymptotic independence it suffices to check the following 2 conditions:

$$\int_{[0,1]^2} R_n(x,y)W(x,y)\mathrm{d}x\mathrm{d}y \to 0 \quad \text{and} \quad \int_{[0,1]^2} \left(\int_0^1 R_n(x,z)W(z,y)\mathrm{d}z \right)^2 \mathrm{d}x\mathrm{d}y \to 0. \tag{B.1}$$ eq:WR

The first condition in (B.1) follows from [38, Lemma 8.22]. For the second condition note that:

$$\begin{aligned}
&\int_{[0,1]^2} \left(\int_0^1 R_n(x,z)W(z,y)\mathrm{d}z \right)^2 \mathrm{d}x\mathrm{d}y \\
&= \int_{[0,1]^4} R_n(x,z)W(z,y)R_n(x,z')W(z',y)\mathrm{d}z\mathrm{d}z'\mathrm{d}x\mathrm{d}y \\
&= \int_{[0,1]^2} \left(\int_{[0,1]^2} g_n(x,y)R_n(x,z)W(z,y)\mathrm{d}x\mathrm{d}z \right) \mathrm{d}y \quad \left(\text{where } g_n(x,y) := \textstyle\int_{[0,1]} R_n(x,z')W(z',y)\mathrm{d}z'\right) \\
&= \|R_n\|_\square \to 0.
\end{aligned}$$

This establishes the second condition in (B.1) and completes the proof of Lemma B.1. □

Appendix C. A CLT for Weighted Sum of Centered χ^2 Random Variables

In this section we establish a Central Limit Theorem for an infinite weighted sum of independent centered χ^2 random variables.

m:chisquareCLT⟩ **Lemma C.1.** *Consider a sequence of infinite sequences $\{(a_s^{(n)} : s \geq 1)\}_{n\geq 1}$ satisfying the following conditions:*

$$\lim_{n\to\infty} \max_{s\geq 1} |a_s^{(n)}| = 0 \text{ and } \lim_{n\to\infty} \sum_{s\geq 1} (a_s^{(n)})^2 = \tau^2, \tag{C.1}$$ eq:sequence

for some constant $\tau \geq 0$*. Let* $\{\xi_s\}_{s\geq 1}$ *be an i.i.d. sequence of* $\chi^2_k - k$ *random variables. Then the infinite sum* $\sum_{s\geq 1} a_s^{(n)} \xi_s$ *is well defined, for* n *large enough, and, as* $n \to \infty$*,*

$$\sum_{s\geq 1} a_s^{(n)} \xi_s \xrightarrow{D} N\left(0, 2k\tau^2\right). \tag{C.2}$$
eq:chisquaredsu

Proof. Fix $M \geq 1$ and let $\mathcal{F}_M := \sigma(\{\xi_s\}_{s=1}^M)$ be the sigma algebra generated by $\{\xi_s\}_{s=1}^M$ and $S_M^{(n)} := \sum_{s=1}^M a_s^{(n)} \xi_s$. Then $(S_M, \mathcal{F}_M)$ is a martingale with

$$\sup_{M\geq 1} \mathbb{E}(S_M^{(n)})^2 \leq \sum_{s\geq 1} (a_s^{(n)})^2 < \infty,$$

for all n large enough. Hence, the sum $\sum_{s\geq 1} a_s^{(n)} \xi_s$ is well defined, for n large enough.

We will show (C.2) by establishing the convergence of the Moment Generating Function (MGF). By [10, Proposition 7.1] we know that the MGF of $\sum_{s\geq 1} a_s^{(n)} \xi_i$ is well defined in a neighborhood of zero. In particular, for $|\lambda| < \frac{1}{8}$, one has

$$\mathcal{E}_n(\lambda) := \mathbb{E}\left(e^{\lambda \sum_{s\geq 1} a_s^{(n)} \xi_i}\right) = \prod_{s\geq 1} \frac{e^{-\lambda k a_s^{(n)}}}{\left(1 - 2\lambda a_s^{(n)}\right)^{\frac{k}{2}}}.$$

Taking logarithms on both sides,

$$\begin{aligned}
\log \mathcal{E}_n(\lambda) = -\frac{k}{2} \sum_{s\geq 1} \log\left(1 - 2\lambda a_s^{(n)}\right) - \lambda k \sum_{s\geq 1} a_s^{(n)} &= \frac{k}{2} \sum_{s\geq 1} \sum_{t=1}^{\infty} \frac{(2\lambda a_s^{(n)})^t}{t} - \lambda k \sum_{s\geq 1} a_s^{(n)} \\
&= \frac{k}{2} \sum_{s\geq 1} \sum_{t=2}^{\infty} \frac{(2ta_s^{(n)})^t}{t} \\
&= \frac{k}{2} \sum_{s\geq 1} \sum_{t=3}^{\infty} \frac{(2\lambda a_s^{(n)})^t}{t} + k\lambda^2 \sum_{s\geq 1} (a_s^{(n)})^2. \qquad (C.3)
\end{aligned}$$
eq:mgf_doublesu

Denote by $a_s^{(n)}$ the s-th largest among $\{a_s^{(n)}\}_{s\geq 1}$. Then, for any $L \geq 1$, we have

$$4\tau^2 \geq \sum_{s\geq 1} (a_s^{(n)})^2 \geq \sum_{s=1}^{L} (a_{(s)}^{(n)})^2 \geq L(a_{(L)}^{(n)})^2,$$

for all n large enough. (Note that the first inequality in the display above holds for all n large enough by the second condition in (C.1).) This implies, $a_{(L)}^{(n)} \leq \frac{2\tau}{\sqrt{L}}$, for n large enough and $L \geq 1$. Hence, for $|\lambda| < \frac{1}{8\tau}$,

$$\sum_{s\geq 1} \sum_{t=3}^{\infty} \frac{|\lambda a_s^{(n)}|^t}{t} \leq \sum_{s\geq 1} \sum_{t=3}^{\infty} |\lambda a_{(s)}^{(n)}|^t \leq \sum_{s\geq 1} \sum_{t=3}^{\infty} \left|\frac{4\lambda\tau}{\sqrt{s}}\right|^t \leq \sum_{s\geq 1} \sum_{t=3}^{\infty} \left|\frac{1}{2\sqrt{s}}\right|^t < \infty. \tag{C.4}$$
eq:ansum

This shows that the double sum in (C.3) is absolutely convergent. Also, for $t \geq 3$,

$$\lim_{n\to\infty} \left|\sum_{s\geq 1} (a_s^{(n)})^t\right| \leq \lim_{n\to\infty} \max_{s\geq 1} |a_s^{(n)}| \sum_{s\geq 1} (a_s^{(n)})^2 = 0,$$

from (C.1). Hence, by the Dominated Convergence Theorem (observe that the bound in (C.4) is uniform over n),

$$\lim_{n\to\infty} \sum_{s\geq 1} \sum_{t=3}^{\infty} \frac{|\lambda a_s^{(n)}|^t}{t} = 0.$$

Hence, taking limits as $n \to \infty$ on both sides of (C.3) gives, for $|\lambda| < \frac{1}{8\tau}$

$$\lim_{n\to\infty} \mathcal{E}_n(\lambda) = e^{k\tau^2\lambda^2},$$

which is the MGF of $N(0, 2k\tau^2)$. This completes the proof of Lemma C.1. □

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